B.TECH. TMA - 011

SIXTH SEMESTER EXAMINATION, 2007-2008

GRAPH THEORY

Time : 3 Hours *Total Marks : 100*

Note. Attempt ALL questions.

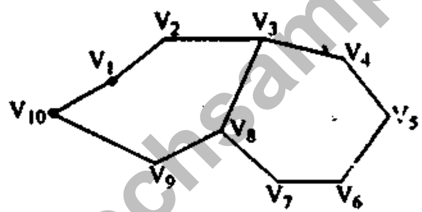
Q.1. Answer any FOUR of the following:

(5×4=20)

Q.1. (a) Define the degree of a vertx in a graph. Prove that the number of vertices of odd degree in a graph is always even.

Ans. Euler graph: A closed trial containing all the edges without repeating is called euler graph.

Hamiltonean graph: A Hamiltonion graph n a connected graph G defined by a closed trail in which traverses every vertex extactly once. G_1 graph is not possible.



G_2 : Hamiltonian non eulerian graph.

1. **(b) Prove that in a graph with n vertices and k componnts the max. number of edges cannot exceed $(n - k)(n - k + 1)/2$.**

1. **(c) Define an eulerian and a hamiltonian graph. Give examples of eulerian nonhamilton graph G and hamiltonian non-eulerian graph G_2 with No of vertices ≥ 10.**

1. **(d) Define a connected graph. prove that for a graph withe xactly two vertices of odd degree, there must be a path joining these two vertices.**

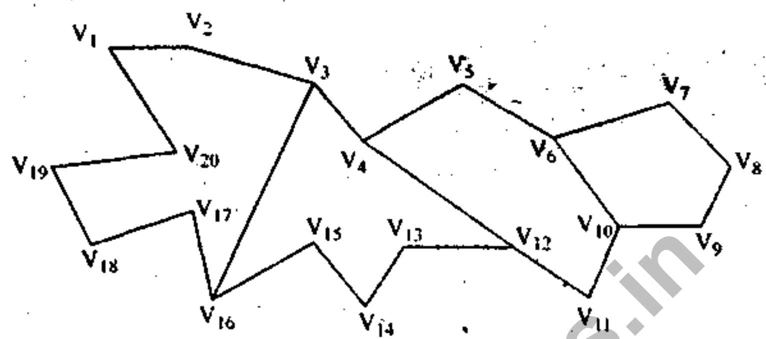
Ans Connected graph: A graph is said to be connected if there is at least one path from every pair of vertices.

Let G be a graph with all even vertices except vertices V_1 and V_2 which are odd.

By hand shaking lemma, every component of a diaconnected graph, no graph can have an odd number of odd vertices. If V_1 and V_2 belong to different components then each one of these two components shall have one vertex of odd degree which is not possible. Hence in G, V_1 and V_2 must belong to the same components and must have a path between them.

1. **(e) Draw a graph G with a hamiltonian path but without a hamiltonian circuit with No. of vertices ≥ 20.**

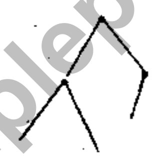
Ans.



1. **(f) Define a tree. Prove that a graph with *n* vertices, *n* – 1 edges, and no circuits is connected.**

Ans. Tree: A tree is a cyclic connected graph.

For example



Continue to page 4.4

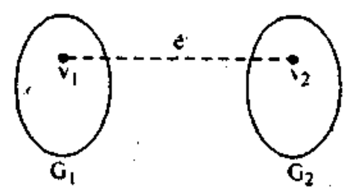
A graph G with n vertices and n – 1 edges and no cycles is connected.

Proof: Let us suppose that G is a disconnected acyclic graph with n vertices and n – 1 edges. It will consist of two or more acyclic components. Suppose G consists of two acyclic components G_1 and G_2.

Now we add an edge e between a vertex v_1 in G_1 and another vertex v_2 in G_2 which makes the G as connected.

As there was no path between the vertices v_1 and v_2 in G (bring vertices of two acyclic components) addition of the edge e does not create and cycle. Therefore by addition of e, the graph is now a connected acyclic graph i.e. a tree with n verties and n edges which is impossible by virtue of theorem 4.3. Thus there is a contradiction and therefore the graph G which is acyclic with n vertices and n – 1 edges cannot be disconnected, it is a connected graph.

Hence the theorem



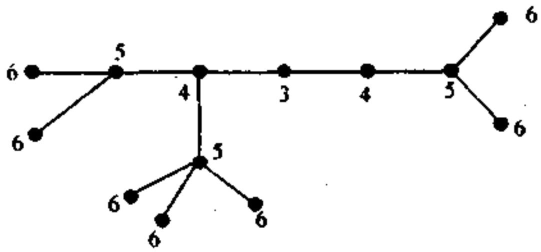
Q.2. Attempt any four of the following: **(5×4=20)**

2. **(a) Prove that every tree has one or two centres.**

Ans. Proof: Suppose T is a tree with more than two vertices. It must have two or more pendant vertices.

Let us delete all thependant vertices of the tree T. Then the resulting graph T_1 shall also be a tree as deletion of pendant vertices does not affect the connectedness and acyclic character of T.

Moreover, by deleting all the pendant vertices inT, the eccentricities of each of the remaining vertices in T (i.e. eahc of the vertices in T_1) is reduced by one. Therefore, the vertices that were centres in T_1 shall remain centres in T_1. If we continue this process as shown below for a tree T. we shal be left either with a single vertex (which is centre of T) or with an edge (whose end vertices are centres of T). The same vertices shall be the centres of T_1 also.



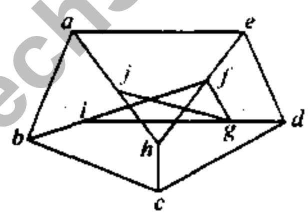
Stage-1

Stage-2

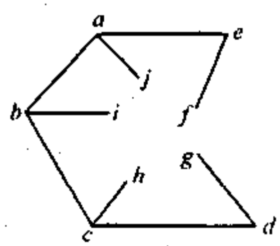
Stage-3

Stage-4

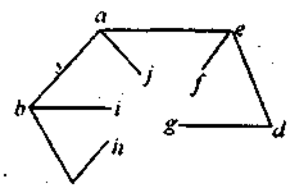
2. (*b*) **Define a spanning tree of a graph. Find four spanning trees of the following. Peterson's graph.**



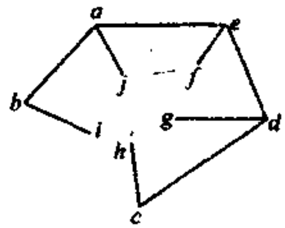
Ans. Spanning Tree: A acyclic subgraph of a connected graph G. Which contains all the vertices of G, is called spanning tree T of G. The fair spanning tree T_1, T_2, T_3 and T_4 are following from given graph.



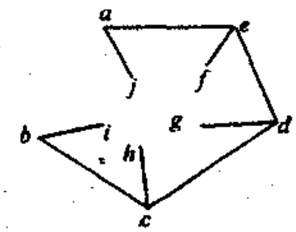
T_1



T_2



T_3



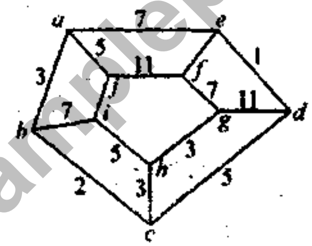
T_4

2. **(c) Prove that w.r.t any of its spanning trees a conneced graph with n vertices and e edges has $n - 1$ tree branches and $e - n + 1$ chords.**

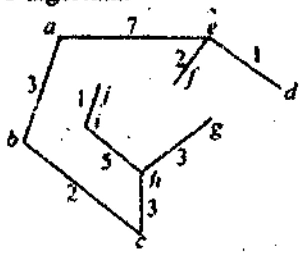
Ans. In a connected graph G with n vertices and e edges has $(n - 1)$ tree brances then the chards of a connected graph is defined by the Total number of edges of graph spanning tree edges.

Since the graph is connected having n-vertices and edges so the spanning tree edges are $n - 1$ by definition of spanning tree. So chords are $e - (n - 1)$ i.e. $e - n + 1$.

2. **(d) Find a shortest spanning tree in a weighted-graph G using the PRIM's algorithm where G is as follows:**

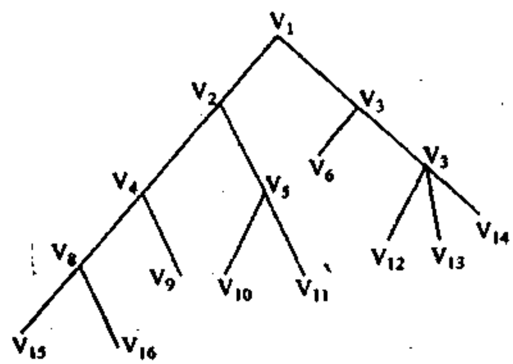


Ans. Spaning Tree using PRIM's algorithm



2. **(e) Construct a tree with 16 vertices, each corresponding to a spanning tree of a labelled completed graph with four vertices.**

Ans.

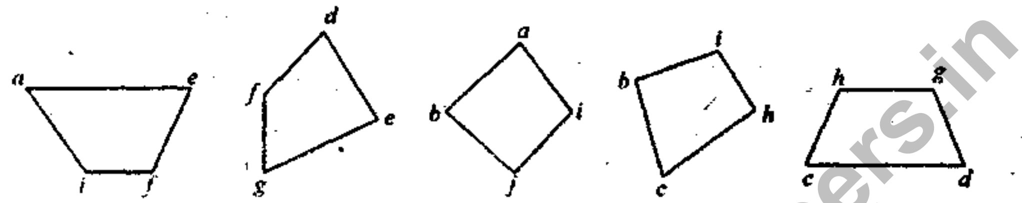


2. (*f*) **Define fundamental cicuit and cut sets. Find five fundamental circuits fundamental cut setso of the graph in question 2(b).**

Ans. Fundamental circuit: A closed cicuitthat is obtained by adding any one chord of a spanning tree T in a connecte graph is called fundamental circuit.

Cut-set: It is a minimal set of edges in connecte graph whose removal makes the graph disconnected.

The five fundamental circuits are:



The five cut-set are:

{(*a*, *b*), (*a*, *f*), (*a*, *e*)}

{(*b*, *a*), (*b*, *i*), (*b*, *c*)}

{(*c*, *h*), (*c*, *b*), (*c*, *d*)}

{(*d*, *c*), (*d*, *g*), (*d*, *e*)}

{(*e*, *d*), (*e*, *f*), (*e*, *a*)}

Q.3. Attempt any four of the following: **(5×4=20)**

(*a*) **Define the vertex connectivity and edge connectivity of a graph. Prove that for a graph G with *n* vertices and *e* edges vertex connectivity ≤ edge connectivity ≤ $\frac{2e}{n}$.**

Ans. Vertix connectivity: It is a minimal number of vertices whose removal from graph G leaves the remaining graph disconnected.

Edge connectivity: It is a minimum number of edges whose removal disconnects the graph.

The given graph G with n vertix and edges. The total 2e degree is divided among n vertices. Therefore, there msut be at last one vertex in G whose degree is equal to or less than the number 2e/e. Thus the vertex connectivity of G cannot exceed this number. We know that any tree is minimal connected graph.

i.e. $e \geq n - 1$

we have, vertex connectivity ≤ edge connectivity ≤ $\frac{2e}{n}$.

So, minimum vertex connectivity = $\frac{2e}{n}$.

3. (*b*) **Define the capacity of a cut-set. Prove that the maximum flow possible between two vertices *a* and *b* in a network is equal to the minimum of capacities of all cut-sets with respect to *a* and *b*.**

Ans. Capacity of a cat-set: It is defined as the sum of capacities of all those edges in the net work that are directed rom the vertices in set P to the vertices in set $\bar{P}$. It is denoted by

$C(P, \bar{P})$.

Thus $\quad C(P, \bar{P}) = \sum_{\substack{i \in P \\ j \in \bar{P}}} C_{ij}$

Suppose $(P, \bar{P})$ is any cut of a network such that the source at P and $b \in \bar{P}$ then

$$\sum_j f_{ji} - \sum_i f_{ij} = 0 \qquad ...(1)$$

This is true for all theintermediary vertices in the network G. Therefore summing equation (1) for all the intermediary vertices j contained in the subset P of vertices of the cut $(P, \bar{P})$

$$\sum_{j \in P}\left[\sum_i f_{ji} - \sum_i f_{ij}\right] = 0 \qquad ...(2)$$

where except a j denotes any intermediate vertices $(P - I)$ we also have

$$\sum_i f_{ai} - \sum f_{ia} = \omega \qquad ...(3)$$

Total flow through all the edges incident out of the source - total flow through all the edges incident into the source = ω.

3. (c) Define a separable graph. Prove that a non-separable graph G set of edges incident on each vertex of G is a cut-set.

Ans. Separable graph: Separable graphs are those graphs whose vertex connectivity is one or if there exists a subgraph H in a connected graph G such that there is only one vertex common in H and its complement $\bar{H}$ is then G is said to be separable graph. It is a 1-connected graph.

3. (d) Define a planar graph. Prove that a complete graph with five vertices in non-planar.

Ans. Please see Q.No. 3(*b*) of 2004-05.

3. (e) For a planar gaph with n vertices and e edges, prove that $e \leq 3n - 6$.

Ans. Please see Q.No. 3(*a*) of 2004-05.

3. (f) Define thickness and crossing number of a graph. Find thickness and crossing numbers of the graphs K_5 and K_{3}, 3.

Ans. Thickness of a graph: The minimum number of a planar subgraphs whose union is the given graph G, is called the thickness of G.

Crossing number: It is the minimum number o crossing in the drawing of the geometrical representation fo a graph in the plane. The thickness of k_5 and $k_{3,3}$ is two the crossing number of k_5 and $k_{3,3}$ are two and six.

Q.4. Attempt any two of the following: **(2×10=20)**

4. (a) Define a vector space of graph. Find five base and number of vectors in the vector space of graph of question 2(*d*). Also, find five cut-set vectors and ive circuit vectors of this vector space.

Ans. Let G be a graph with four vertices and five edges e_1, e_2, e_3, e_4 and e_5. Any subset of these five edges of G can be represented by a 5-type

$$X = (x_1, x_2, x_3, x_4, x_5)$$

such that $$x_i = \begin{cases} 1 & \text{if } e_i \text{ is in } G \\ 0 & \text{if } e_i \text{ is not in } G \end{cases}$$

4. (*b*) Define the adjacency matrix of a graph. Find the nrank of the reuglar graph with *n* vertices and with degree $p(< n)$ of verty vertex.

Ans. Adjacency matrix: Let G be a graph with n vertices and no parallel edges. It can be represented by an n × n symmetric binary matrix.

$$X = [x_{ij}]$$

Such that $x_{ij} = 1$ if there is an edge between *i*th and *j*th vertices v_i and v_j respectively.

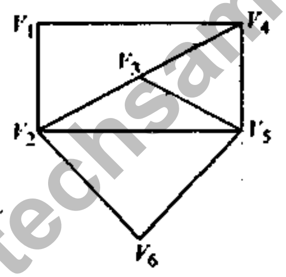
The rank of regular graph with degree p is $p - 1$.

4. (*c*) Define redcuced matrix A_f, fundamental circuit matrix b_f and fundamental cut-set matrix C_f of a connected graph *G* with *n* vertices and *e* edges. Derive the relationships among A_f, B_f and C_f.

Ans. Please see Q.No. 4(*b*) of 2002-03.

Q.5. Attempt any two of the following: (2×10=20)

5. (*a*) Define the chromatic polynomial of a graph. Find the chromatic polyunomial of the graph given below:



Ans. Please see Q.No. 5(*a*) of 2002-03.

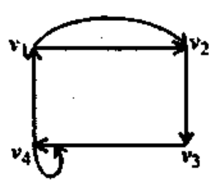
5. (*b*) State and prove five colour theorem.

Ans. Please see Q.No. 5(*b*) of 2002-03.

5. (*c*) Define directed graph (diagraph), simple digraph, asymmetric digraph, symmetric digraph, complete symmetric digraph and complete asymmetric digraph. Give example in each case. Also, prove that of theincidence matrix A(G) of digraph G, determinant of every square submatrix of A(G) is 1, – 1 or 0.

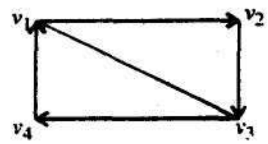
Ans. Diagraph: Digraph consists of set o vertices V and set of edges E. Each edges is associated with ordered pair of vertices:

Example:



Simple diagraph: A digraph without parallel edges or self loop is called simple digraph.

Example:



Assymmetric digraph: It is digraph that has at the most oné directed edge betwene a pair of vertices but is allowed to hae a self loop.

Symmetric digraph: It is a graph having any pair of vertices have two directed edges if connected.

Complete symmetric digraph: It is a simple graph in which there si exactly one edge directed from every vertex to every other vertex.