B.TECH.

ECS-505

SIXTH SEMESTER EXAMINATION 2010-11

GRAPH THEORY

Note: *Q. No. 1 is compulsory and carriers 5 marks. Attempt one question from each unit symbols have their usual meaning.*

Q 1. Attempt any four parts of the following :

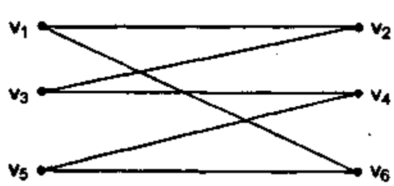
(3 × 4 = 12)

(a) What is a bipartite graph? Obtain expression for the maximum number of edges in bipartite graph.

Ans. Bipartite Graph: A graph $G = (V, E)$ is said to be bipartite if its vertex set V can be partitioned into two non empty disjoint subsets V_1 and V_2 such that every edge is set $E(G)$ has one end vertex in the set V_1 and the other end vertex in the set V_2. No edge of G can join two vertices of the same subset V_1 or V_2.

Complete Bipartite graph, denoted by $k_{m,n}$ isa graph whose vertex set V is partitioned into two disjoint subsets V_1 and V_2 consisting of m and n vertices respectively. Each edge of the graph connects vertex of V_1 to a vertex of V_2 and every vertex of V_1 is a adjacent to every vertex of V_2.

Example: A cycle C_6 is bipartite graph, because its vertex set V can be partitioned into two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and also every edge in C_6 join a vertex in set V_1 to a vertex in set V_2 as shown below in Fig.



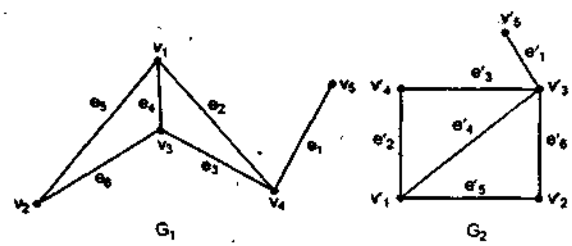
A expression to obtain maximum no. of edge in bipartite graph is $(m + n)$.

(b) When are two graph said to be isomorphic? Show that two graphs need not be isomorphic even they both have the same order and same size.

Ans. Two graphs G_1 and G_2 are said to be isomorphic to each other if there is a one to one correspondence between their vertices and between their edges so that the incidence relationship is maintained.

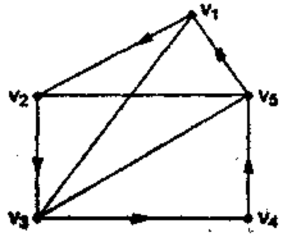
It means that if in graph G_1 an edge e_k is incident with vertices vi and vj then in graph G_2 its corresponding edge e_k must be incident with the vertices v_i and v_j that correspondent to the vertices v_i and v_j respectively.

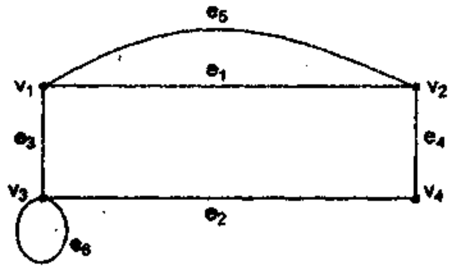
The following two graphs G_1 and G_2 are not isomorphic graphs.



(c) Define the Hamiltonian graph. Draw a graph that has a Hamiltonian circuit.

Ans. Hamiltonian Graph: It is a graph G that contains a Hamiltonian cycle. A Hamiltonian cycle is a closed trail (or circuit) containing every vertex of the graph G exactly once except the initial vertex which is also the terminating vertex. The following graphs are Hamiltonian cycle shown through arrows.



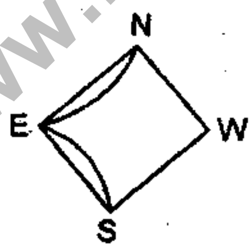


(d) Discuss the travelling salesman problem.

Ans. Please See Q. 1 (d) of fifth Semester Examination 2010-11.

(e) In a park, jogging track is designed in such a way that there are four end points (say N, E, W, S). End point W is connected by two paths from end points N and S each and by single path from end point E. End points N and E are connected by single path. End points S and E are also connected by single path. Show that a jogging person can't return to its starting end point after walking through all the paths exactly once.

Ans.



(f) Prove that if a connected graph G is decomposed into two subgraphs g_1 and g_2, there must be at least one vertex common between g_1 and g_2.

Ans. It a connected graph G is decomposed into two subgraph g_1 and g_2 then the graph will be disconnected. Since a graph is connected it means there is at least one path between every pair of vertices. If we decompose the connected graph by vertex decomposion then all the path of graph G will pass through that vertex or set of vertices. When we **decompose the graph G the basic property of connected graph will remain unchanged.** That is whatever path will be made by graph G the same path will also be make by graph g_1 and g_2 and this is possible only if the decomposion vertex or set of vertices is common in both subgraph g_1 and g_2. So that there must be at least one vertex common between g_1 and g_2.

Q. 2. Attempt any two parts of the following:

(6 × 2 = 12)

(a) Show that:

(i) If a graph G have one and only one path between every pair of vertices. G is tree.

Ans. There is one and only on path between every pair of vertices of a tree.

Proof: (a)As a tree T is defined as a connected graph, there must exist at least one path between every pair vertices of T.

Existence of more than one (two or more) distinct paths between any two vertices v1 and v2 of T shall create a cycle in T which is not acceptable for a tree.

(b) Hence there can not be two or more than two paths between any pair of vertices of a tree. By virtue of (a) and (b) there is one and only path between every pair of vertices of a tree.

(ii) The number of terminal vertices in a binary tree with n vertices is $(n + 1)/2$.

Ans. p, the number of pendant vertices in a binary tree T with n vertices $= \frac{n+1}{2}$.

Proof: Total number of vertices in the given binary tree $T = n$

Number of pendant vertices in $T = p$

Number of vertices with degree 2 in $T = 1$

Therefore, number of vertices with degree 3 $n - (p + 1) = n - p - 1$

Total number of degrees of all vertices in T.

= number of degrees of p vertices each of degree 1.

+ number of degree of 1 vertex of degree 2

+ number of degrees of $n - p - 1$ vertices each of degree 3.

$= (p \times 1) + (1 \times 2) + \{(n - p - 1) \times 3]$

$= 3n - 2p - 1$ ...(1)

Alternatively, the number of edges in the tree T with n vertices = $n - 1$

The total number of degrees associated with these $n - 1$ edges (and hence n vertices)

$= 2(n - 1) = 2n - 2$ ... (2)

From (1) and (2)

$3n - 2p - 1 = 2n - 2$

or $n - 2p + 1 = 0$ ... (3)

or $$p = \frac{n+1}{2}$$

(b) (i) Define the terms : Metric and Fundamental circuit.

Ans. (i) A function f (x, y) ot two variables x and y satisfying the following conditions is called a metric.

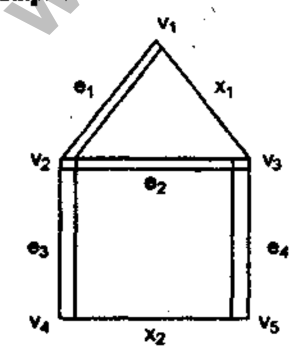
1. Non-negativity $f(x, y) > 0$ and $f(x, y) = 0$ if and only if $x = y$.

2. Symmetry: $f(x, y) = f(x, y)$

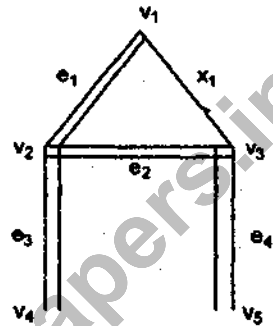
3. Triangular inequality: $f(x, y) \leq f(x, z) + f(z, y)$ for any z.

Fundamental circuit: A closed circuit or cycle that is obtained by adding any one chord of a spanning tree T in a connected graph is called a fundamental circuit.

For Example:

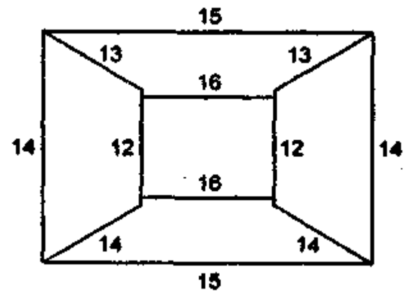


In a above graph G_1, e_1, e_2, e_3, e_4 are edges of spanning tree which are represented by double line and x_1 and x_2 are are chord of spanning tree. It we add either x_1 or x_2 in spanning free T we get a fundamental circuit i.e., e_1, e_2, e_3, e_4, x_1

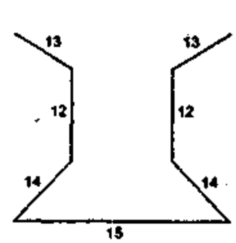


(ii) Prove that the nullity of a graph does not change when you either insert a vertex in the middle of an edge, or remove a vertex of degree two by merging two edges incident on it.

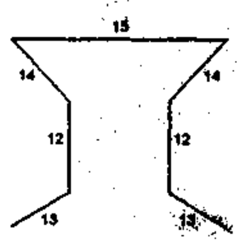
(c) Find all the minimum spanning trees in the following graph using Prin's algorithm:



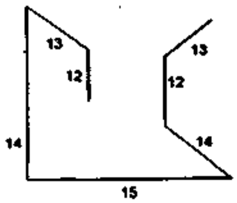
Ans. A given is weighted graph and its all minimum spanning trees are



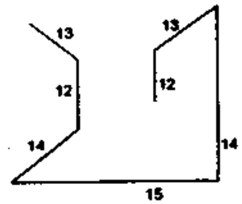
(i)



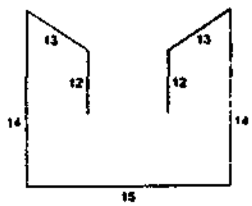
(ii)



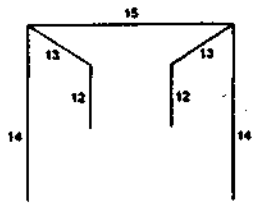
(iii)



(iv)



(v)



(vi)

Q 3. Attempt any two parts of the following:

(a) Define the edge-connectivity and vertex connectivity of a graph. Prove that for any graph.

$k(G) \leq \lambda(G) \leq \delta(G)$.

Where $k(G)$, $\lambda(G)$, $\delta(G)$ are connectivity number, edge connectivity number and minimum degree among the vertices in a graph respectively.

Edge Connectivity

$\lambda(G)$ – Edge connectivity of a graph G is the minimum number of edges whose removal disconnects it. It is denoted by $\lambda(G)$.

Example : $\lambda(G)$ in fig. 3 is 1

White $\lambda(G)$ in Fig. 4 is 2

We may observe that in a certain graph G, the removal of different sets of edges may make the graph disconnected. In such a case, we shall be interested in finding out the set having minimum number of edges that disconnects it.

Proof: Let v be a vertex with least degree $d(v)$ in G..

The edge connectivity or the minimum number of edges to disconnects the vertex v from the rest of the graph = $d(v)$ as to disconnect any other vertex or vertices from the rest of the graph we shall have to delete all the edges incident on those vertices which is always greater or equal to $d(v)$ as it is the least degree.

Now, we shall prove a theorem concerned with this property of ring sum of two cut-sets.

Hence, the edge connectivity of $G \leq d(v)$.

Vertex connectivity of a connected graph G

It is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected. It is clear that removal of any vertex also removes the edge. if any, incident with that vertex.

Note: Vertex connectivity is meaningful only if the graph has three or more vertices and is not complete.

(b) Describe an algorithm to detect the palnarity of a graph

Detect palnarity of $K_{3,3}$.

Ans. Please see Q. 3 (b) of 2004-05

(c) Define the thickness and crossing number of a graph. Find the thickness and cross in number of the complete graph with n vertices, where $n \leq 8$.

Ans. The minimum number of Planar subgraphs whose union is the given graph G, is called the thickness of G.

It leads to the conclusion that the thickness of Planar graph is one.

Some other observations:

1. Thickness of each of the two kuratowski's graphs is 2.

2. Thickness of a complete graph of 9 vertices is 3.

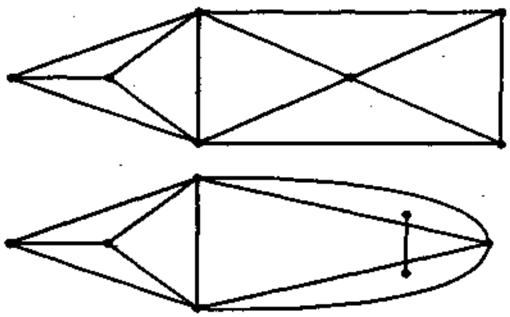
Crossing number

It is the minimum number of crossing or intersection in the drawing of the geometrical representation of a graph in the plane.

It is a measure of deviation of the given graph from Planarity. Therefore, the crossing number of a Planar graph is zero.

The crossing number of each of the two Kuratowsk's graphs is one.

Example. Showthat the graph in the following fig. are isomorphic, but their geometric duals are not isomorphic.



Q 4. Attempt any four parts of the following:

(a) Prove that the ring sum of two circuits in a graph G is either a circuit or an edge-disjoint union of circuits.

The ring sum of any two circuits in a graph G is either a circuit or edge disjoint union of circuits.

Proof: Suppose C_1 and C_2 are any two circuits in graph G. There may be two cases:

Case 1: The two circuits C_1 and C_2 have no edges or vertices in common.

In this case the rind sum $C_1 \oplus C_2$ shall be a disconnected sub-graph of G which will be an edge - disjoint union of circuit.

Case 2: The two circuits C_1 and C_2 have edges or vertices or both edges and vertices in common.

In this case let us consider a vertex v in G. We know that degree of every vertex in a circuits C_1 or C_2 is two. Also, $v \in C_1$ or $v \in C_2 \Rightarrow v$ Î $C_1 \oplus C_2$.

(i) Now, if $v \in C_2$ only $\bar{v} \in C_2$ only or if one of the edges formerly incident on v was in both the circuit C_1 and C_2, then $d(v) = 2$ (even).

(ii) If C_1 and C_2 intersect at v without having a common edge, the $d(v) = 2$ (even).

Therefore, degree of each vertex in $C_1 \oplus C_2$ is even. So $C_1 \oplus C_2$ is an Euler graph which will. Hence the theorem has proved.

(b) Define the cut set subspace of connected graph. What is meant by dimesion of a subspace?

Ans. The set of all cut-set vectors in vector space V_G is called cut-set subspace of connected graph.

The dimension of a subspace is defined by the total number of linearly independent vector necessary to span the subspace.

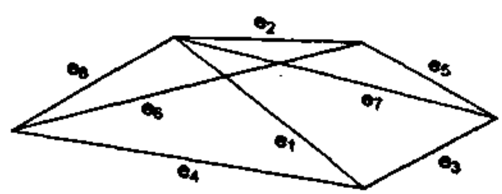
We know that for connected graph G with n vertices and edges the nullity is $e - n + 1$. That is it is equal to the total no. of edges minus total no. of vertices plus one. If we add one edge and one vertices or if we remove one edge and one vertex the nullity will not change. So in a connected graph G if we either insert a vertex in the middle of an edge or remove a vertex of degree two by merging two edges incident on it, the nullity will not change.

(c) Define a circuit vector and a cut set vector of a connected graph. Prove that a circuit vector and a cut set vector are orthogonal to each other w.r.t mod 2 arithmetic.

Ans. Plese see Q. 4 (c) of fifth semester examination 2010-11.

(d) Sketch a graph G that has the following vector (among others) in its circuit subspace: (0, 1, 1, 1, 1, 0, 0, 1), (0, 1, 1, 1, 0, 1, 1, 0), (0, 1, 0, 0, 1, 0, 1, 0), (0, 1, 0, 0, 0, 1, 0, 1), (1, 0, 1, 0, 1, 1, 0, 1), (1, 0, 1, 0, 0, 0, 1, 0), (1, 0, 0, 1, 1, 1, 1, 0) and (1, 0, 0, 1, 0, 0, 0, 1).

Ans. A graph G corresponding to given condition is shown below



(e) Find Chromatic polynomial $P(G, x)$, where G is a cyclic graph with n vertices when $n = 3$ or $n = 4$.

Ans. A chromatic polynomial $P(G, x)$ where G is cyclic graph with $n = 3$ is

$$P_n(x) = x(x - 1)$$

And $P(G, x)$ with $n = 4$ is

$$P_n(x) = x(x - 1)(x - 2)$$

(f) Explain the covering and partitioning of a graph.

Ans. Partitioning of a graph. A graph G is aid to be have been partitioned into two subgraphs G_1 and G_2. It

$$G_1 \cup G_2 = G$$

and $$G_1 \cap G_2 = \text{Null graph}$$

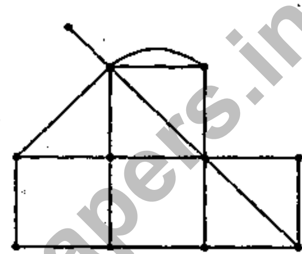
i.e., every edges of G occurs either in G1 or in G2 but not in both.

An ***edge covering*** or a ***covering subgraph*** or a covering of graph G is defined as a set H of edges in G such that every vertex in G is incident on at least one edge in H.

This set of edges H is said to cover the graph G. Every graph has its own covering.

Minimal Covering G: It is a covering of G such that removal of any edge or edges from H destroys the property of it's covering the graph G.

Example: The set of edges shown in dark lines is the minimal covering of the given graph.



Covering Number: The number of edges in a minimal covering of the smallest size (containing minimum number of edges) is called the covering number of the graph. [A graph can have several minimal coverings.

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