B. Tech.

SIXTH SEMESTER EXAMINATION, 2009-10

Graph Theory

(TMA-011)

Time : 3 Hours **Total Marks : 100**

Note : (i) Attempt all the questions.

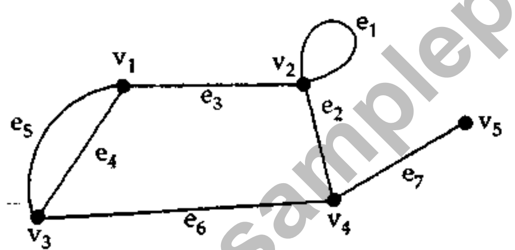
(ii) All questions carry equal marks.

Q.1 Attempt any four parts of the following :

(a) Define the degree of a vertex in a graph. Prove that the sum of the degrees of all vertices of a graph in a graph is twice the number of edges in graph.

Ans. The number of edges incident on a vertex V_i, with self-loops counted twice, is called the degree, $d(V_i)$ of vertex V_i.

For example



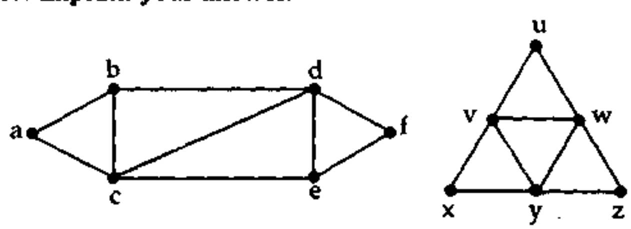
$$d(V_1) = d(V_3) = d(V_4) = 3, d(V_2) = 4 \text{ and } d(V_5) = 1$$

The degree of a vertex is sometime also referred to as its valency.

Let us now consider a graph G with e edges and n vertices $V_1, V_2, \ldots, V_n$. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. That is,

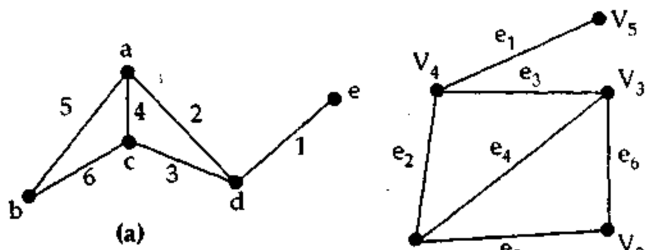
$$\sum_{i=1}^{n} d(V_i) = 2e$$

(b) Define isomorphism of graphs. For the following pair of graphs, determine whether or not the graphs are isomorphic. Explain your answer.



Ans. Two graphs G and G' are said to be isomorphic (to each other) if there is a one to one correspondence between their vertices and between their edges such that edge e is incidence relationship is preserved.

In other words, suppose that edge e is incident on vertices V_1 and V_2 in G ; then the corresponding edge e' in G' must be incident on the vertices V'_1 and V'_2 that correspond to V_1 and V_2 respectively. For example, one can verify that the two graphs in the below figure are isomorphic.



(b)
Isomorphic graphs

The correspondence between the two graphs is as follows: The vertices a, b, c, d and e correspond to V_1, V_2, V_3, V_4 and V_5 respectively. The edges 1, 2, 3, 4, 5 and 6 correspond to e_1, e_2, e_3, e_4, e_5 and e_6 respectively.

Given graphs are not isomorphic there are three vertices of degree 4 in figure two but in figure one there are only two vertices of degree 4.

(c) Prove that a simple graph with n vertices and k components can have atmost $(n-k)(n-k+1)/2$ edges.

Ans. Let G is a graph

Let the number of vertices in each of the k components of G be $n_1, n_2, \ldots\ldots n_k$

Thus we have

$$n_1 + n_2 + n_3 + \ldots\ldots n_k = n$$

$$n_i \geq 1$$

the proof of the theorem depends on an algebric inequality

$$\sum_{i=1}^{K} n_i^2 \leq n^2 - (k-1)(2n-k) \quad \ldots(1)$$

Now the maximum number of edges in the i^{th} component of G (which is simple connected graph) is $\frac{1}{2} n_i(n_i - 1)$.

$\therefore$ max. number of edges in G is

$$\frac{1}{2}\sum_{i=1}^{K}(n_i = 1)n_i \frac{1}{2}\left(\sum_{i=1}^{K} n_i^2\right) - \frac{n}{2}$$

$$\boxed{\geq \frac{1}{2}[n^2 - (k-1)(2n-k)] - \frac{n}{2} \text{ from (1)}}$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

Hence Proved

(d) Discuss travelling salesman Problem.

Ans. This problem is closely related to the Hamiltonian circuits. A salesman is required to visit a number of cities during a trip. Given the distances between the cities, in what order should be travel so as to visit every city precisely one and return home, with the minimum mileage travelled.

Representing the cities by vertices and the roads between them by edges, we get a graph. In this graph, with every edges e_i there is associated a real number (the distance in miles) $w(e_i)$. Such a graph is called a weight of edge e_i.

If each of the cities has read to every other city, we have a complete weighted graph. This graph has numerous Hamiltonian circuits, we are to pick the one that has the smallest sum of distances (or weights) Hamiltonian circuits in a complete graph of n vertices can be shown to be (n – 1)!/2. This follows from the fact that starting from the first vertex, (n – 2) from the second (n – 3) from the third and so on. These being independent choices, we get (n – 1)! possible number of choices. This number is, however, divided by 2, because each hamiltonian circuit has been counted twice. Theoretically, the problem of the travelling all salesman can always be solved by enumerating (n – 1)!/2. Hamiltonian circuits, calculating the distance travelled in each, and then picking the shortest one.

(e) Define the following with one example.

(i) Complete graph

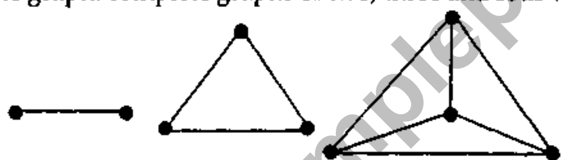
(ii) Eulerian graph

(iii) Hamiltonian graph

(iv) Bi-partite graph

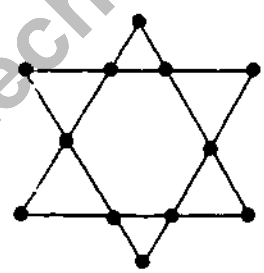
(v) Cut points of a graph

Ans. (I) Complete graph : A simple graph in which there exists an edge between every pair of vertices is called a complete graph. complete graphs of two, three and four vertices are shown.



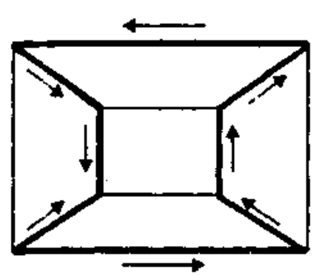
(ii) Eulerian graph : If some closed walk in a graph contains all the edges of the graph, then the walk is called an Euler line and the graph and Eulerian graph.

For example :

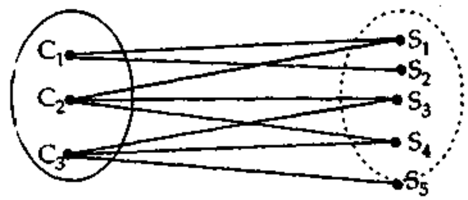


(iii) Hamiltonian graph : A Hamiltonian graph in a connected graph is defined as a closed walk that traverse every vertex of G exactly once, except of course the starting vertex, at which the walk also terminates.

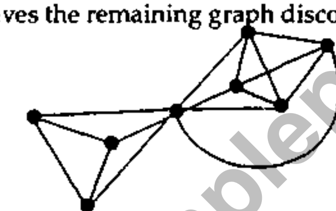
For example



(iv) Bi-partite graph : A graph G is called bi-partite if its vertex set V can be decomposed into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex in V_1. This graph can have no self-loop. A set of parallel edges between a pair of vertices can all be replaced with one edge with affecting bi partiteness of a graph.



(v) Cut-points of a graph : On examining the graph, we find that although removal of no single edge (or even a pair of edges) disconnects the graph, the removal of the single vertex V does. Therefore, we define the term called cut point of a graph. The cut point of a graph G is defined as the minimum number of vertices whose removal of G leaves the remaining graph disconnected.



Q.2. Attempt any four parts of the following :

(a) If G is a non-trivial tree, then prove that G contains at least two vertices of degree 1.

Ans. You must have observed that each of the trees shown in the figures has several pendant vertices (a pendant vertex was defined as a vertex of degree one). The reason is that in a tree of n vertices we have n – 1 edges, and hence 2(n – 1) degrees to be divided among n vertices. Since no vertex can be of zero degree, we must have atleast two vertices of degree one in a tree. This of course makes sense only if $n \geq 2$.

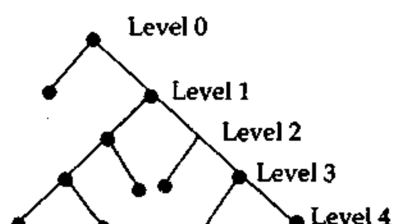
(b) Define binary trees and discuss two important applications of it.

Ans. A binary tree is defined as a tree in which there is exactly one vertex of degree two, and each of the remaining vertices is of degree one or three.

Since, the vertex of degree two is distinct from all other vertices, this vertex serve as a root. Thus every binary tree is a rooted tree.

Two important applications are :

(1) The number of vertices n in a binary tree is always odd. This is because there is exactly one vertex of even degree, and the remaining (n – 1) vertices are of odd degrees. Since, the number of vertices of odd degrees is even, (n – 1) is even Hence n is odd



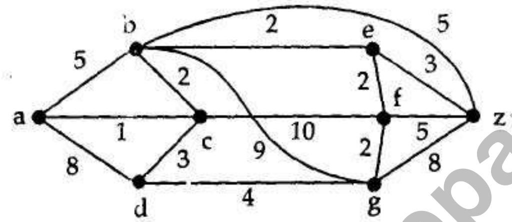
4 level binary tree

(2) Let P be the number of pendant vertices in a binary tree T. Then (n – P – 1) is the number of vertices of degree three. Therefore, the number of edges in T equals.

$$\frac{1}{2}[P+3(n-P-1)+2]=n-1$$

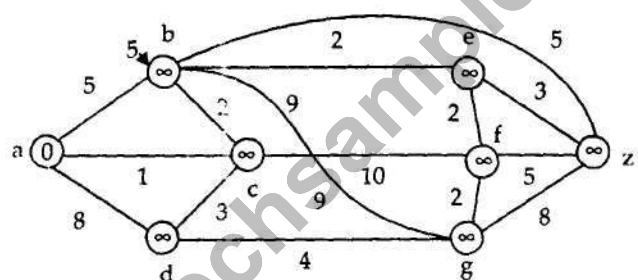
hence $$P=\frac{n+1}{2}$$

(c) Apply Dijkstra algorithm to find out the shortest path from the vertices a to z in the following graph.



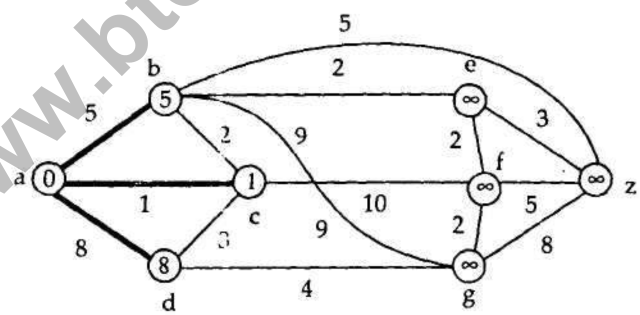
Ans.

(i)

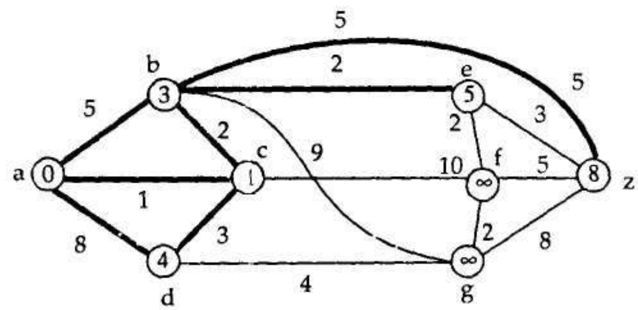


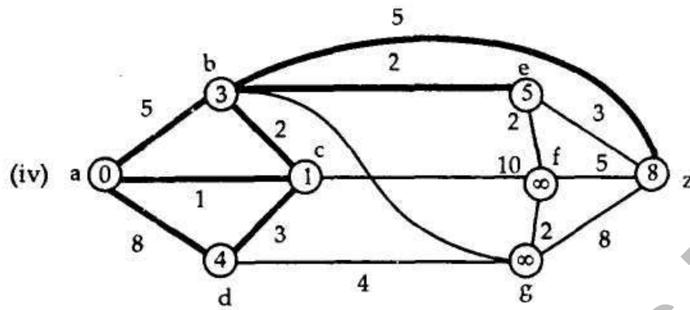
a as a source noe

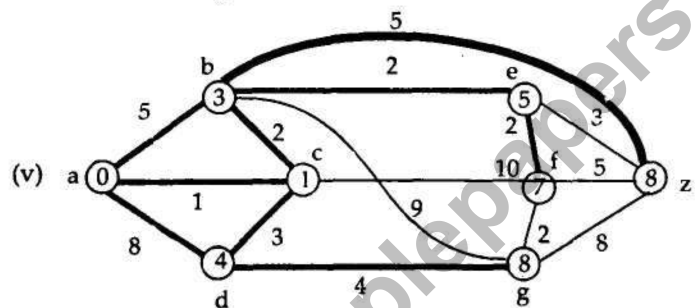
(ii)



(iii)

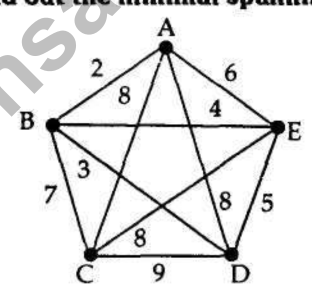






∴ Total cost of the graph is 0-1+3+4+5+7+8+8=36

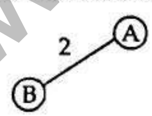
(d) Use Prims algorithm to find out the minimal spanning tree of the following graph.



Ans. Starting from vertex A

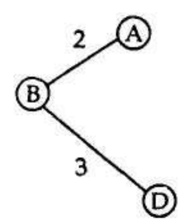
But don't take close loop

(i)



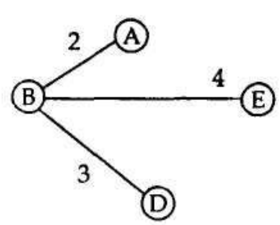
min (*AB, AE, AC, AD*) = *AB*

(ii)

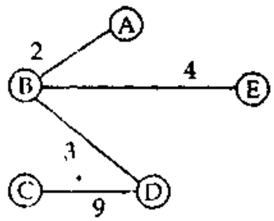


min (*BE, BD, BC*) = *BD*

(iii)



min (*BA, BE, BD, BC*) = *BE*

(iv) 

$\min(DC, DF) = DC$

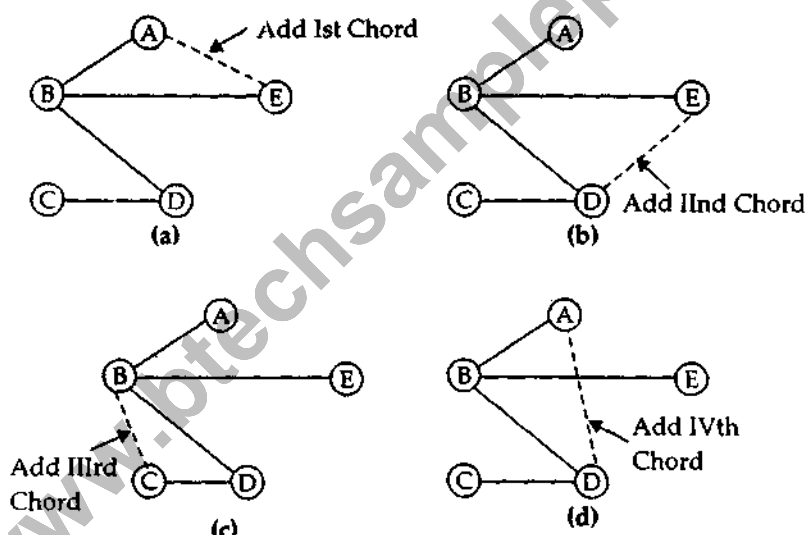
$\therefore$ Total cost $= 2 + 4 + 3 + 9$

$= 18$

(e) Define fundamental circuit. Find the sets of fundamental circuits (four only) of the graph given in Q.No. 2(d). Take any spanning tree and find it corresponding to that spanning tree.

Ans. Consider a spanning tree T in a given connected graph G. Let C_i be a chord with respect to T, and let the fundamental circuit made by C_i be called Γ, consisting of K branches $b_1, b_1, \ldots b_k$ in addition thto the chard C_i; that is $\Gamma = \{C_i, b_1, b_2, b_3, \ldots\ldots, b_k\}$ is a fundamental circuit with respect to T.

Now, according to Question the sets of fundamental circuits (four only) of the graph given in Q.No. 2(d) are



(f) Define eccentricity of the vertex and centre of a graph. Find the centre of the graph given in question no. 2(d).

Ans. The eccentricity E(V) of a vertex V in a graph G is the distance from V to the vertex farthest from V in G, that is,

$$E(V) = \max_{V_i \in G} d(V, V_i)$$

A vertex with minimum ecentricity in graph G is called a centre of G.

Now, according to Question there is no centre in the graph 2(d) because there is no pendant vertex or degree 1 vertex.

Q.3. Attempt any four parts of the following :

(a) Define a planar graph. State and prove the Euler's formula for planar graph.

Ans. A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect. Note that the *"Meeting"* of edges at a vertex is not considered an intersection. A graph that cannot be drawn on a plane without a Crossover between its edges is called non planer.

Euler formula $\qquad f = e - n + 2$

here, $(e - n + 2)$ are the regions in the graph

Proof : It will suffice to prove the theorem for a simple graph, because adding a self-loop or a parallel edge simply adds one region to the graph and simultaneously increases the value of e by one. We can also disregard (i.e, remove) all edges that do not form boundaries and any region. Addition (or removal) of any such edge increases (or decreases) e by one and increases (or decreases) n by one, keeping the quantity $e - n$ unaltered.

Since any simple planar graph can have a plane representation such that each edge is a straight line, any planar graph can be drawn such that each region is a polygon (a polygonal net). Let the polygonal net representing the given graph consist of f regions or faces, and let k_p be the number of p-sided regions. Since each edge is on the boundary of exactly two regions,

$$3.k_3 + 4.k_4 + 5.k_5 + \ldots + r.k_r = 2.e, \qquad \ldots(1)$$

where k_r is the number of polygons, with maximum edges

Also,

$$k_3 + k_4 + k_5 + \ldots\ldots + k_r = f. \qquad \ldots(2)$$

The sum of all angles subtended at each vertex in the polygonal net is

$$2\pi n \qquad \ldots(3)$$

Recalling that the sum of all interior angles of a p-sided polygon is $\pi(p-2)$, and the sum of the exterior angles in $\pi(p+2)$, let us compute the expression in (3) as the grand sum of all interior angles of $f - 1$ finite regions plus the sum of the exterior angles of the polygon defining the infinite region. This sum is

$$\pi(3-2).k_3 + \pi(4-2).k_4 + \ldots + \pi(r-2).k_r + 4\pi$$
$$= \pi(2e - 2f) + 4\pi \qquad \ldots(4)$$

Equating (4) to (3), we get

$$2\pi(e - f) + 4\pi = 2\pi n$$

or

$$e - f + 2 = n$$

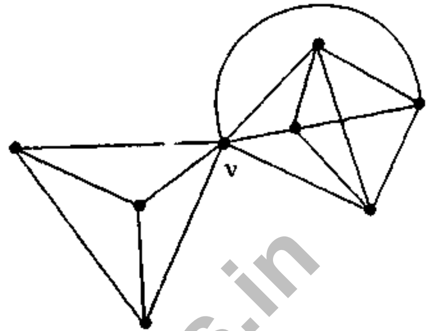
Therefore, the number of regions is

$$f = e - n + 2$$

(b) Define edge and vertex connectivity of a graph. Prove that the vertex connectivity of any graph will never be more than the edge connectivity.

Ans. Edge Connectivity : Each cut-set of a connected graph G consists of a certain number of edges. The number of edges in the smallest cut-set (i.e., cut-set with fewest number of edges) is defined as the edge connectivity of G. Equivalently, the edge connectivity of a connected graph can be defined as the minimum number of edges whose removal (i.e., deletion) reduces the rank of the graph by one. The edge connectivity of a tree, for instance, is one. The edge connectivities of the graphs in Figs. is three.

Vertex Connectivity : On examining the graph in Fig. we find that although removal of no single edge (or even a pair of edges) disconnects the graph, the removal of the single vertex v does. Therefore, we define another analogous term called **vertex connectivity**. The vertex connectivity (or simply connectivity) of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected.



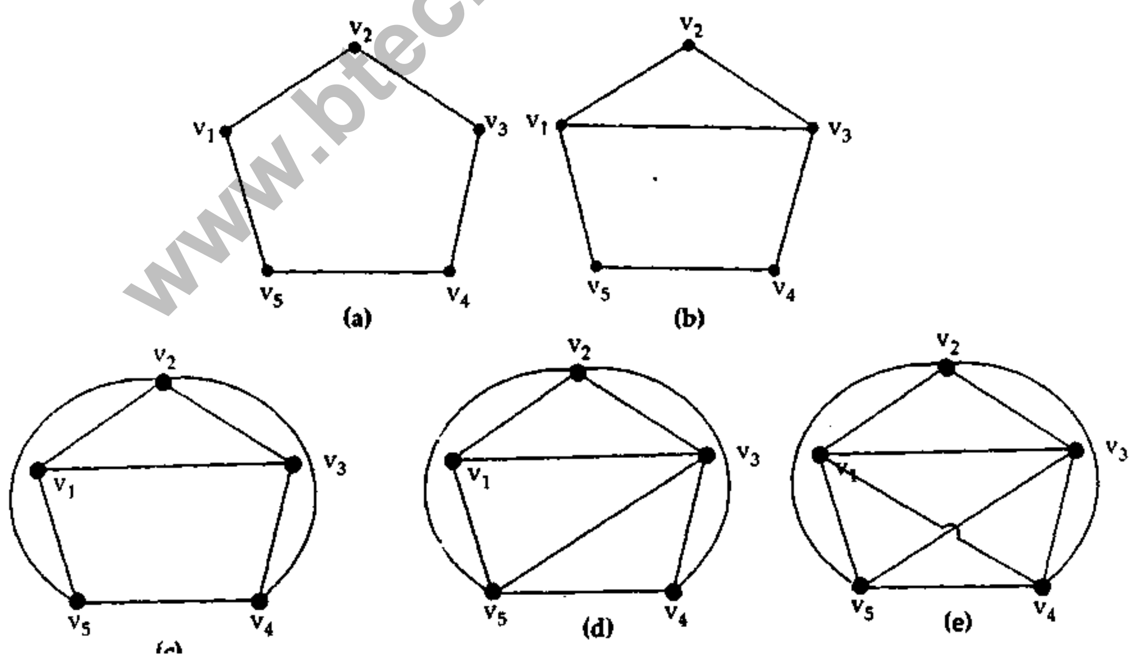
Let vertex V_i be the vertex with the smallest degree in G. Let $d(V_i)$ be the degree of V_i vertex V_i can be separated from G by removing the $d(V_i)$ edges incident on vertex V_i.

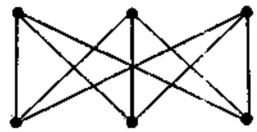
(c) Show that the kuratowski's first (K_5) and second ($K_{3,3}$) are nonplanar graphs.

Ans. Proof : Let the five vertices in the complete graph be named v_1, v_2, v_3, v_4, and v_5. A complete graph, as you may recall, is a simple graph in which every vertex is joined to every other vertex by means of an edge. This being the case, we must have a circuit going from v_1 to v_2 to v_3 to v_4 to v_5 to v_1 – that is, a pentagon. See Fig. (a). This pentagon must divide the plane of the paper into two regions, one inside and the other outside (Jordan curve theorem).

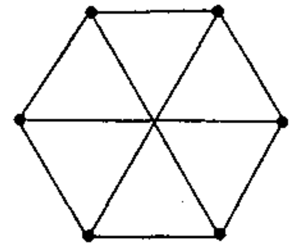
Since vertex v_1 is to be connected to v_3 by means of an edge, this edge may be drawn inside or outside the pentagon (without intersecting the five edges drawn previously). Suppose that we choose to draw a line from v_1 to v_3 inside the pentagon. See Fig. (b). (If we choose outside, we end up with the same argument.)

Now we have to draw an edge from v_2 to v_4 and another one from v_2 to v_5. Since neither of these edges can be drawn inside the pentagon without crossing over the edge already drawn, we draw both these edges outside the pentagon. See fig (c). The edge connecting v_3 and v_5 cannot be drawn outside the





(f)



(g)

pentagon without crossing the edge between v_2 and v_4. Therefore, v_3 and v_5 have to be connected with an edge inside the pentagon. See Fig. (d)

Now we have yet to draw an edge between v_1 and v_4. This edge cannot be placed inside or outside the pentagon without a crossover. Thus the graph cannot be embedded in a plane. See Fig. (e)

A complete graph with five vertices is the first of the two graphs of Kuratowski. The second graph of Kuratowski is a regular connected graph with six vertices and nine edges, shown in its two common geometric representations in Figs. (f) and (g) where it is fairly easy to see that the graphs are isomorphic.

Employing visual geometric arguments similar to those used in proving Theorem (Kuratowski's First graph is non planar), it can be shown that the second graph of Kuratowski is also nonplanar.

(d) Show that a graph has a dual if and only if it is planar.

Ans. We need to prove just the 'only if' part i.e. we have to, only prove that a non-planar graph does not have a dual.

Let G be a non-planar graph. Then according to Kuratowski's theorem, 'G' contains K_5 or $K_{3,3}$ or a graph homeomorphic to either of these. We can see that a graph 'G' can have a dual only if every subgraph 'g' of G and every graph homeomorphic to 'g' has a dual. Thus if we can show that neither K_5 nor $K_{3,3}$ has a dual, we are done.

This we shall show by contradiction as follows :

Suppose $K_{3,3}$ has a dual D. According to the property of dual, we must have that; since $K_{3,3}$ have

(i) 6 vertices (ii) 9 edges (iii) no cutsets of two edges.

Thus D must have

(i) 9 edges (ii) 6 regions (iii) all circuits of length four or six.

∴ Degree of every vertex is atleast four.

D must hae at least 5 vertices each of degree four or more

∴ D must have atleast $(5 \times 4)/2 = 10$

Which is a contradiction

Thus $K_{3,3}$ cannot have a dual.

Likewise, suppose that the graph K_5, has a dual H.

Now K_5 has (i) 10 edges (ii) no pair of parallel edges (iii) no cut-sets with two edges (iv) cut-sets with only four or six edges.

∴ H must have (i) 10 edges (ii) no vertex with degree less than 3 (iii) no pair of parallel edges and (iv) Circuits of length four and six only.

Now graph H contains a Hexagon (a circuit of length six), and no more than 3 edges can be added to a hexagon without creating a circuit of length three or a pair of parallel edges. Since both of these are forbidden in H and H has 10 edges, there must be atleast seven vertices in H. The degree of each of these vertices is atleast three. This leads H to have at least $(7 \times 3)/2 = 10.5$ i.e. 11 edges (contradiction)

Thus K_5 cannot have a dual.

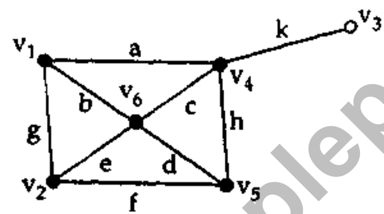
(e) Define the thickness of a graph, give one example. Find the thickness of Kuratowski's first and second graphs.

Ans. This is found that a given graph G is non planar, it is natural to ask, what is the minimum number of planes, necessary for embedding G ? The least number of planar subgraphs whose union is the given graph G is called the **thickness.**

In a printed-circuit board, for instance, the number of insulation layers necessary is the thickness of the corresponding graph.

By definition, then the thickness of a planar graph is one. The thickness of each of Kuratowski's graphs is clearly two. The reader can show, by sketching them, that the thickness of the complete graph of eight vertices is two, while the thickness of the complete graph of nine vertices is three.

(f) Define cut-sets. List all cut-sets with respect to the vertex pair v_2, v_3 in the following graph.



Ans. In a connected graph G, a cut-set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of these edges disconnects G.

A cut-set always "cuts" a graph into two. Therefore, a cut-set can also be defined as a minimal set of edges in a connected graph whose removal reduces the rank of the graph by one.

Now according to question, find cut-set of v_2, v_3

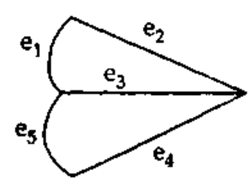
$$v_2 = \{g, e, f\}$$
$$v_3 = \{K\}$$

Q.4. Attempt any two parts of the following :

(a) Define basis vectors of a graph. Show that the number of distinct basis possible in a cut-set subspace is :

$$\frac{1}{r!}(2^r - 2^0)(2^r - 2^1)(2^r - 2^2)........(2^r - 2^{r-1})$$

Ans. If every vector in a vector space W can be expressed as a linear combination of a given set of vectors, this set is said to span the vector space W.



Any set of K linearly independent vectors that spans W, a K-dimensional vector space is called a basis for the vector space W, where dimension of the vector space is the minimal number of linearly independent vectors required to span W corresponding to each subgraph of G there is a vector in W_G, represented by an e-tuple. The natural basis for this vector space W_G is a set of e linearly independent vectors, each representing a subgraph consisting of one edge of G.

The above graph shows the set of the following five vectors serves as a basis of W_G :

(1,0,0,0,0) ;

(0,1,0,0,0) ;

(0,0,1,0,0) ;

(0,0,0,1,0) ;

(0,0,0,0,1) ;

Any of the possible 32 subgraphs (including G as well as the null graph) can be represented by a suitable linear combination of these five basis vectors.

(b) Find the relationship among reduced incidence matrix A_f, fundamental circuit matrix B_f and fundamental cut set matrix C_f of a connected graph. Also establish the relation by giving one example.

Ans. Relationships among A_f, B_f and C_f:

$$B_f = [I_\mu \vdots B_t] \quad ...(1)$$

$$C_f = [C_c \vdots 1_{n-1}] \quad ...(2)$$

Where, subscript 't' = submatrix corresponding to the branches of a spanning tree.

Subscript 'c' = The submatrix corresponding to the chords.

Let the spanning tree T in Eqs (1) and (2) be the same, and let the order of the edges in both equations be the same. Furthermore, in the reduced incident matrix A_f — of size (n – 1) by e-let the edge (i.e. the columns) by arranged in the same order as in B_f and C_f.

Partition A_f into two sub-matrices :

$$A_f = [A_c \vdots A_t] \quad ...(3)$$

where, A_t = consist of the $n-1$ column corresponding to the branches of the spanning tree T.

A_c = Remaining submatrix corresponding to the $e - n + 1$ chords.

Since the column in A_f and B_f are arranged in the same order, From eq.

$$A \,.\, B^T = B \,.\, A^T = 0$$

we have :

$$A_f \,.\, B_f^T = 0 .$$

That is $[A_c : A_t] \,.\, \begin{bmatrix} I_\mu \\ \hline B_t^T \end{bmatrix} = 0,$

and $$A_c + A_t \,.\, B_t^T = 0 \quad ...(4)$$

Since A_t is non-singular, its inverse A_t^{-1} exists. Premultiply both sides of Eq.(4) by A_t^{-1}, we get

$$A_t^{-1} \,.\, A_c = -B_t^T \quad ...(5)$$

Since in mode 2 arithmetic – 1 = 1,

$$B_t^T = A_t^{-1} \,.\, A_c \quad ...(6)$$

Similarly, since the columns in B_f and C_f are arranged in the same order.

$$C_f \,.\, B_f^T = 0$$

That is $[C_c : I_{n-1}] \,.\, \begin{bmatrix} I_\mu \\ \hline B_t^T \end{bmatrix} = 0,$

$$C_c = -B_t^T \quad ...(7)$$

$$= B_t^T \quad ...(8)$$

$$= A_t^{-1} . A_c \text{ from (6)}$$

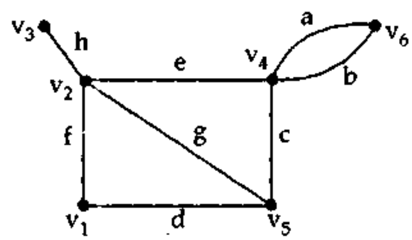
This leads to three conclusions :

1. Given A or A_f, we can readily construct B_f and C_f, starting from an arbitrary spanning tree and its subgraph. A_t in A_f.

2. Given either B_f or C_f, we can construct the other. Thus since B_f determine a graph within 2-isomorphism so does C_f.

3. Given either B_f or C_f, A_f in general cannot be determined completely.

for example :



using $\{a, e, f, g, h\}$ as the spanning tree

$$A_f = \begin{bmatrix} b & c & d & | & a & e & f & g & h \\ 0 & 0 & 1 & | & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= [A_c : A_t]$$

$$B_f = \begin{bmatrix} b & c & d & | & a & e & f & g & h \\ 1 & 0 & 0 & | & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

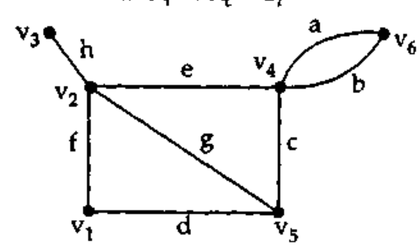
$$= [I_u : B_t]$$

$$C_f = \begin{bmatrix} b & c & d & | & a & e & f & g & h \\ 1 & 0 & 0 & | & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [C_C : I_5]$$

$B_t^T = C_c$ is immediate. It can also be readily verified that $A_t^{-1} \cdot A_c = B_t^T$

Q.(c) (i) If B is a circuit matrix of a connected graph G, with e edges and n vertices, then show that the rank of B is equal to the nullity of G.



Ans. The above graph has four different circuits {a, b}, {c, e, g}, {d, f, g} and {c, d, f, e,}. Therefore, its circuit matrix is a 4 by 8 (0, 1) matrix as shown

$$B(G)=\begin{array}{c}\\1\\2\\3\\4\end{array}\begin{bmatrix} a & b & c & d & e & f & g & h \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Now, according to Question

rank $r = n - K$

nullity $\mu = e - n + K$

Here n = no of vertices,

e = edges,

K = no of components

∴ rank of the above graph

$$r = n - k$$
$$= 6 - 2$$
$$= 4$$

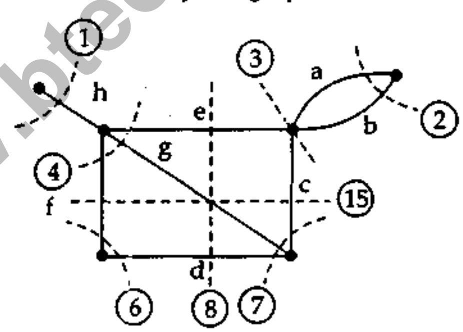
Now, nullity of the above graph

$$\mu = e - n + K$$
$$= 8 - 6 + 2$$
$$= 4$$

Hence proved i.e. rank of a connected graph is equal to nullity of the connected graph.

(c) (ii) Prove that the rank of a cut-set matrix is equal to the rank of the graph.

Ans. This can be proved with the help of a graph i.e.



cut-set graph

therefore, cut-set matrix of a graph is

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}\begin{bmatrix} a & b & c & d & e & f & g & h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

After solving matrix, rank of the matrix is equal to 4

Now, rank of the graph

rank $\quad r = n - k$

here $\quad$ n = no. of nodes

$\quad$ k = no. of components

$\therefore \quad r = n - k$

$r = 6 - 2$

$r = 4$ **Hence Proved.**

Q.5. Attempt any two parts of the following :

(a) Prove that an m-vertex graph is a tree if and only if its chromatic polynomial is :

$$P_m(\lambda) = \lambda(\lambda - 1)^{m-1}$$

Ans. Let G be a tree of m vertices we shall proceed by induction method. If $m - 1$, then G contains only one vertex which can be coloured only in n distinct ways. Thus result is true for m = 1.

If m = 2, then G contains one edge, so that exactly two colors are required for the proper colouring of the graph. Hence $C_1 = 0$ and two colors can be assigned in two different ways for the vertices of the graph therefore $C_2 = 2$ Thus $P_m(\lambda) = 0 + \lambda(\lambda - 1) = \lambda(\lambda - 1)$

Hence the result holds for m = 2. Let us suppose that the chromatic polynomial of a tree of $(\lambda - 1)$ vertices is given by $P_{m-1}(\lambda) = (\lambda - 1)^{m-2}.n$

Since the graph G is a tree T_m, it contains a pendant vertex V. Let $T_{m-1} = T_{m-v}$ then, T_{m-1} is a tree of $(m - 1)$ vertices that mean T_{m-1} be a graph obtained by deleting the vertex V from T_m.

Then by inductive hypothesis the chromatic polynomial of T_{m-1} is $\lambda(\lambda - 1)^{m-2}$

Now we want to prove that a tree of m vertices have chromatic polynomial given by $P_m(\lambda) = \lambda(\lambda - 1)^{m-1}$

Let V' be the neighbours of V in T_{m-1} and colored by any one of n colors i.e. V can be colored by any one of the remaining $(\lambda - 1)$ colors for each proper coloring of T_{m-1}. Hence the number of different ways of properly coloring of T_m is

$$P_m(\lambda) = \lambda(\lambda - 1)^{m-2}.(\lambda - 1) = \lambda(\lambda - 1)^{m-1}$$

So, $\lambda(\lambda - 1)^{m-1}$ ways can properly color the given tree. Thus the result is true by induction hypothesis.

(b) Show that the number of simple labelled graph of n vertices is $2^{n(n-1)/2}$.

Ans. The numbers of simple graphs of n vertices and 0, 1, 2, ..., n (n – 1)/2 edges are obtained by substituting 0, 1, 2, ...n (n – 1)/2 for e in expression

$$\binom{\frac{n(n-1)}{2}}{e}$$

The sum of all such numbers is the number of all simple graphs with n vertices. Then the use of the following identity proves the theorem.

$$\binom{k}{0}+\binom{k}{1}+\binom{k}{2}+.....+\binom{k}{k-1}+\binom{k}{k}=2^k$$

(c) Define indegree and outdegree of a vertex of a directed graph. Prove that for a directed graph D with n vertices $(v_1, v_2, ...v_n)$ and q edges.

$$\sum_{i=1}^{n} \text{in degree}(v_i) = \sum_{i=1}^{n} \text{outdegree}(v_i) = q$$

Ans. In degree and outdegree

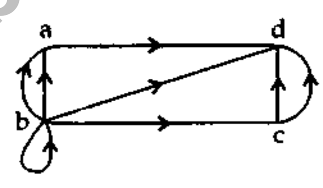
In a directed graph G, the outdegree of a vertex V of G, denoted by $\text{outdeg}_G(V)$ or $\deg_G+(V)$, is the number of edges beginning at V and the indegree of V, denoted by $\text{indeg}_G(V)$ or $\deg_G-(V)$ is the number of edges ending at V. The sum at the indegree and outdegree of a vertex is called the total degree of the vertex. A vertex with zero in degree is called a source and vertex with zero out degree is called sink.

Proof : $\sum_{i=1}^{n} \text{in degree}(v_i) = \sum_{i=1}^{n} \text{outdegree}(v_i) = q$

i.e. the sum of the outdegrees of the vertices of a diagraph G equals the sum of in degree of the vertices which equals the number of edges in G .

Any directed edge (u, v) contributes 1 to the indegree of u and 1 to the out degree of V. Further, a loop at v contributes 1 to the indegree and 1 to the outdegree of V, hence the proof.

Example :



$\text{Indeg}_G(a)=2$, $\text{Indeg}_G(b)=1$

$\text{Indeg}_G(c)=2$, $\text{Indeg}_G(d)=1$

$\text{Outdeg}_G(a)=1$, $\text{Outdeg}_G(b)=1$

$\text{Outdeg}_G(c)=1$, $\text{Outdeg}_G(d)=1$

Note that, the sum of the indegree and sum of outdegree each equal to 8, the number of edges.