B. Tech.

(SEM. I) EXAMINATION, 2007-08

MATHEMATICS-I

Time: **3** *Hours* — *Total Marks:* **100**

Note: *Attempt all the problems. Internal choice are mentioned in every problem.*

Q. 1. Attempt any Two parts : (10×2=20)

(*a*) Define the given values, eigenvectors and the characteristic equation of a square matrix. Find the characteristics equation/polynomial, eigen values and eigen vectors of the matrix :

$$\begin{bmatrix} 2 & 5 & 7 \\ 5 & 3 & 1 \\ 7 & 0 & 2 \end{bmatrix}$$

Ans. Characteristic Equation : Let A be a square matrix. Then the equation

$$|A-\lambda I| = 0$$

is called characteristic equation of matrix A.

Eigen Value : Let A be a square matrix. The values of λ satisfying

$$|A-\lambda I| = 0$$

are called eigen values of matrix A.

Eigen Vectors : Let A be a square matrix. Let λ be an eigen value of A. A non zero column vector x satisfying

$(A-\lambda I)x = 0$ is called eigen vector of matrix A corresponding to eigen value λ.

IInd Part :

$$\text{Given matrix } A = \begin{bmatrix} 2 & 5 & 7 \\ 5 & 3 & 1 \\ 7 & 0 & 2 \end{bmatrix}$$

Characteristics equation of A is

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 5 & 7 \\ 5 & 3-\lambda & 1 \\ 7 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(3-\lambda)(2-\lambda)] - 5[5(2-\lambda)-7] + 7[-7(3-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)(6-3\lambda-2\lambda+\lambda^2) - 5(10-5\lambda-7) + 7(-21+7\lambda) = 0$$

$$\Rightarrow (2-\lambda)(6-5\lambda+\lambda^2) - 5(3-5\lambda) - 147 + 49\lambda = 0$$

$$\Rightarrow 12-10\lambda+2\lambda^2-6\lambda+5\lambda^2-\lambda^3-15+25\lambda - 147+49\lambda = 0$$

$$\Rightarrow -\lambda^3+7\lambda^2+52\lambda-150 = 0$$

$$\Rightarrow \lambda^3-7\lambda^2-52\lambda+150 = 0$$

which is not further solvable at our level

Q.1. (*b*) Check the consistancy of the following system of linear non-homogeneous equations and find the solution, if exists :

$$7x_1 + 2x_2 + 3x_3 = 16$$
$$2x_1 + 11x_2 + 5x_3 = 25$$
$$x_1 + 3x_2 + 4x_3 = 13$$

Ans. Given system of equation is

$$7x_1 + 2x_2 + 3x_3 = 16$$
$$2x_1 + 11x_2 + 5x_3 = 25$$
$$x_1 + 3x_2 + 4x_3 = 13$$

$$\text{Augmented matrix } [A \,.\, B] = \left[\begin{array}{ccc|c} 7 & 2 & 3 & 16 \\ 2 & 11 & 5 & 25 \\ 1 & 3 & 4 & 13 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 13 \\ 2 & 11 & 5 & 25 \\ 7 & 2 & 3 & 16 \end{array}\right] \qquad R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 13 \\ 0 & 5 & -3 & -1 \\ 0 & -19 & -25 & -15 \end{array}\right] \qquad R_2 \to R_2 - 2R_1$$

$$R_3 \to R_3 - 7R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 13 \\ 0 & 20 & -12 & -4 \\ 0 & -19 & -25 & -15 \end{array}\right] \qquad R_2 \to 4R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 13 \\ 0 & 1 & -37 & -79 \\ 0 & -19 & -25 & -75 \end{array}\right] \qquad R_2 \to R_2 + R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 13 \\ 0 & 1 & -37 & -79 \\ 0 & 0 & -7281 & -1576 \end{array}\right] \qquad R_3 \to R_3 + 19R_2$$

Which is in Cohelon form

Here, Rank of $A = 3$

Rank of $[A \,.\, B] = 3$

No. of unknown = 3

: Rank of $[A \,.\, B]$ = Rank of A = No. of unknown

: System has unique soln.

Rewriting the system,

$$1x_1 + 3x_2 + 4x_3 = 13 \qquad \text{...(1)}$$

$$1x_2 - 37x_3 = -79 \qquad \text{...(2)}$$

$$-728x_3 = -1567 \qquad \text{...(3)}$$

From (3),

$$x_3 = \frac{-1576}{-728} = \frac{197}{91}$$

Putting x_3 in (2),

$$x_2 = -79 + 37 \times \frac{197}{91}$$

$$\Rightarrow \qquad x_2 = \frac{100}{91}$$

Putting x_2 and x_3 in (1),

$$x_1 = 13 - 3 \times \frac{100}{91} - 4.\ \frac{197}{91}$$

$$\Rightarrow \qquad x_1 = \frac{95}{91}$$

Thus

$$x_1 = \frac{95}{91},\ x_2 = \frac{100}{91},\ x_3 = \frac{197}{91}$$

Q. 1 (*c*) Find the inverse of the matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{5} \\ 1/3 & 1/5 & 1/7 \\ 1/5 & 1/7 & 1/11 \end{bmatrix}$$

Ans. Given matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{5} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{11} \end{bmatrix}$

$\because \qquad A = IA$

$$\Rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{2}{5} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{11} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad R_1 \to 2\,R_1$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{2}{5} \\ 0 & \frac{-1}{5} & \frac{+1}{105} \\ 0 & \frac{+1}{105} & \frac{3}{275} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ \frac{-2}{3} & 1 & 0 \\ \frac{-2}{5} & 0 & 1 \end{bmatrix} A$$

$$R_2 \to R_2 - \frac{R_1}{3}$$

$$R_3 \to R_3 - \frac{R_1}{5}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{2}{5} \\ 0 & 1 & -\frac{3}{7} \\ 0 & \frac{+1}{105} & \frac{3}{275} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 30 & -45 & 0 \\ \frac{-2}{5} & 0 & 1 \end{bmatrix} A$$

$$R_2 \to 45R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{24}{35} \\ 0 & 1 & -\frac{3}{7} \\ 0 & 0 & \frac{202}{13475} \end{bmatrix} = \begin{bmatrix} -18 & 30 & 0 \\ 30 & -45 & 0 \\ -\frac{24}{35} & \frac{+3}{7} & 1 \end{bmatrix} A$$

$$R_1 \to R_1 - \frac{2}{3}\,R_2$$

$$R_3 \to R_3 - \frac{1}{103}\,R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{24}{35} \\ 0 & 1 & -\frac{3}{7} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -18 & 30 & 0 \\ 30 & -45 & 0 \\ \frac{-4620}{101} & \frac{+5775}{202} & \frac{13475}{202} \end{bmatrix}$$

$$A \ R_3 \to R_3 \times \frac{13475}{202}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1350}{101} & \frac{1050}{101} & \frac{-4620}{101} \\ \frac{1050}{101} & \frac{-6615}{202} & \frac{+5775}{202} \\ -\frac{4620}{101} & \frac{+5775}{202} & \frac{13475}{202} \end{bmatrix} A$$

$$R_1 \to R_2 - \frac{24}{35} R_3$$

$$R_2 \to R_2 + \frac{3}{7} R_3$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1350}{101} & \frac{1050}{101} & \frac{-4620}{101} \\ \frac{1050}{101} & \frac{-6615}{101} & \frac{5775}{202} \\ -\frac{4620}{101} & \frac{5775}{202} & \frac{13475}{202} \end{bmatrix}$$

$$= \frac{1}{202} \begin{bmatrix} 2700 & -13230 & -9240 \\ 2100 & -13230 & 5775 \\ -9240 & 5775 & 13475 \end{bmatrix} \text{Ans.}$$

Q. 2. Attempt any Two parts : (10×2=20)

(a) State Leibnitz theorem for n^{th} differential coefficient of the product of two functions. If $y^{1/m} + y^{-1/m} = 2x$, prove that

$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Ans. Leibnitz's Theorem : For two functions u and v. $(uv)n = nco\, unv + n_{c1}u_{n-1}v_1 + ... + n_{cn}uv_n$

2nd part :

$$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$$

$$\Rightarrow \quad z + z^{-1} = 2x \qquad \text{Put } y^{\frac{1}{m}} = z$$

$$\Rightarrow \quad z^2 + 1 = 2xz$$

$$\Rightarrow \quad z^2 - 2xz + 1 = 0$$

$$\Rightarrow \quad z = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2-1}}{2}$$

$$\Rightarrow \quad z = x \ \sqrt{x^2-1}$$

$$\Rightarrow \quad y^{1/m} = x \pm \sqrt{x^2-1}$$

$$\Rightarrow \quad y = \left(x \pm \sqrt{x^2-1}\right)^m$$

Case - II

When $y = \left(x \pm \sqrt{x^2-1}\right)^m$

$$\Rightarrow \quad y_1 = m\left(x + \sqrt{x^2-1}\right)^{m-1} \left(1 + \frac{2x}{2\sqrt{x^2-1}}\right)$$

$$\Rightarrow \quad \sqrt{x^2-1}\, y_1 = m\left(x + \sqrt{x^2-1}\right)^{m-1} \left(x + \sqrt{x^2-1}\right)$$

$$\Rightarrow \quad \sqrt{x^2-1}\, y_1 = -m\left(x - \sqrt{x^2-1}\right)^m$$

$$\Rightarrow \quad \sqrt{x^2-1}\, y_1 = -my$$

$$\Rightarrow \quad (x^2-1)\, y_1^2 = m^2 y^2$$

Thus in both cases,

$$(x^2-1)\, y_1^2 = m^2 y^2$$

Differentiating above expression.

$$2x\, y_1^2 + (x^2-1) = 2y_1 y_2 = m^2\, 2yy_1$$

$$\Rightarrow \quad xy_1 + (x^2-1)\, y_2 = m^2 y$$

$$\Rightarrow \quad (x^2+1\, y_2 + xy_1 - m^2 y = 0 \qquad ...(1)$$

Differentiating (1) n times by Cebhitz's theorem.

$$(x^2-1)\, y_{n+2} + {}^nC_1(2x)\, y_{n+1} + {}^nC_2(2)\, y_n]$$
$$+ [x\, y_{n+1} + {}^nC_1(1)\, y_n] - m^2 y_n = 0.$$

$$\Rightarrow (x^2-1)\, y_{n+2} + 2nx\, y_{n+1} + n(n-1)\, y_n$$
$$+ xy_{n+1} - xy_n - m^2 y_n = 0$$

$$\left[\because C_1 = n, {}^nC_2 = \frac{n(n-1)}{2}\right]$$

$$\Rightarrow (x^2-1)\, y_{n+2} + (2nx + x)\, y_{n+1}$$
$$+ [n(n-1) + n - m^2]\, y_n = 0$$

$$\Rightarrow (x^2-1)\, y_{n+2} + (2n+1)\, xy_{n+1} + (h^2 + m^2)\, y_n$$
$$= 0 \quad \text{Hence Proved.}$$

Q. 2 (b) Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

where $u(x,y) = \log_e\left(\frac{x^2+y^2}{xy}\right)$

Ans. $u = \log \frac{x^2+y^2}{xy}$

To verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

We shall first calculate $\frac{\partial^2 u}{\partial x \partial y}$

Now $\frac{\partial u}{\partial x}$ $u = \log \frac{x^2+y^2}{xy}$

$\Rightarrow \quad u = \log(x^2+y^2) - \log x - \log y$

$\Rightarrow \quad \frac{\partial u}{\partial y} = \frac{xy}{x^2+y^2} - \frac{1}{y}$

$\Rightarrow \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)$

$= \frac{\partial}{\partial x}\left[\frac{2y}{x^2+y^2} - \frac{1}{y}\right] = \frac{2y(2x)}{-(x^2+y^2)^2}$

$\Rightarrow \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{-4xy}{(x^2+y^2)^2}$ ...(1)

Again, $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[\log(x^2+y^2) - \log x - \log y]$

$= \frac{2x}{x^2+y^2} - \frac{1}{x}$

$\Rightarrow \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

$= \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2} - \frac{1}{x}\right)$

$= -2x\left(\frac{+1}{x^2+y^2}\right)^2 2y$

$\Rightarrow \quad \frac{\partial^2 u}{\partial y \partial x} = -\frac{-4xy}{(x^2+y^2)^2}$ ...(2)

From (1) and (2)

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial y \partial x}$$

Q. 2 (c) If $u = x\sin^{-1}\left(\frac{x}{y}\right) + y\sin^{-1}\left(\frac{y}{x}\right)$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

Ans. $u = x\sin^{-1}\left(\frac{x}{y}\right) + y\sin^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow \quad u(x,y) = x\left[\sin^{-1}\left(\frac{x}{y}\right) + \frac{y}{x}\sin^{-1}\left(\frac{y}{x}\right)\right]$

$= x^0 f\left(\frac{y}{x}\right)$

$\Rightarrow$ u is a homogeneous function of degree $n = 1$

$\therefore$ By Fuler's theorem :

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$ ...(1)

Differentiating (1) partially w.r. to x,

$1 \cdot \frac{\partial}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$

$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$ ...(2)

Again,

Diff. (1) partially w.r. to y,

$\frac{\partial}{\partial y}\left[x\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y\frac{\partial u}{\partial y}\right] = \frac{\partial u}{\partial y}$

$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y}$

$\Rightarrow \quad xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$ ...(3)

Adding (2) and (3)

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$ **Ans.**

Q.3. Attempt any Four parts : **(5×4=20)**

(a) Expand $e^x \cos y$ about the point $\left(1, \frac{\pi}{4}\right)$.

Ans. Given function

$$f(x, y) = e^x \cos y$$
$$\Rightarrow \quad f = e^x \cos y$$
$$\Rightarrow \quad f = -e^x \sin y$$
$$f_{xx} = e^x \cos y$$
$$f_{xy} = -e^x \sin y$$
$$f_{yy} = -e^x \cos y$$
$$f_{xxx} = -e^x \cos y$$
$$f_{xyy} = -e^x \cos y$$
$$f_{yyy} = e^x \sin y$$

Thus we get the following table :

Function	Value : at $\left(1, \frac{\pi}{4}\right)$
f	$\frac{e}{\sqrt{2}}$
f_x	$\frac{e}{\sqrt{2}}$
f_y	$-\frac{e}{\sqrt{2}}$
f_{xx}	$\frac{e}{\sqrt{2}}$
f_{xy}	$-\frac{e}{\sqrt{2}}$
f_{yy}	$-\frac{e}{\sqrt{2}}$
f_{xxx}	$\frac{e}{\sqrt{2}}$
f_{xxy}	$-\frac{e}{\sqrt{2}}$
f_{xyy}	$-\frac{e}{\sqrt{2}}$
f_{yyy}	$\frac{e}{\sqrt{2}}$

$\therefore$ By Taylor's theorem,

$$f(x, y) = f\left(1, \frac{\pi}{4}\right) + \frac{1}{1!}\left[(x-1) f_x + \left(y - \frac{\pi}{4}\right) f_y\right] + \frac{1}{2!}$$
$$\left[(x-1)^2 f_{xx} + 2(x-1)\left(y - \frac{\pi}{4}\right) f_{xy} + \left(y \frac{\pi}{4}\right)^2 f_{yy}\right]$$
$$= \frac{e}{\sqrt{2}} + \frac{1}{1!}\left[(x-1)\frac{e}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right)\right] + \frac{1}{2!}$$
$$\left[\frac{e}{\sqrt{2}}(x-1)^2 - 2\frac{e}{\sqrt{2}}(x-1)\left(y - \frac{\pi}{4}\right) - \frac{e}{\sqrt{2}}\left(y - \frac{\pi}{4}\right)^2\right]$$
$$+ \frac{1}{3!}\left[\frac{e}{\sqrt{2}}(x-1)^3 - 3\frac{e}{\sqrt{2}}(x-1)^2\left(y - \frac{\pi}{4}\right) - \frac{3e}{\sqrt{2}}(x-1)\left(y - \frac{\pi}{4}\right)^2 + \frac{e}{\sqrt{2}}\left(y - \frac{\pi}{4}\right)^3\right]$$
$$= \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y \frac{\pi}{4}\right) + \frac{1}{2}(x-1)^2 - (x-1)\left(y - \frac{\pi}{4}\right) - \frac{1}{2}\left(y - \frac{\pi}{4}\right)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{2}(x-1)\left(y - \frac{\pi}{4}\right)^2 + \frac{1}{6}\left(y - \frac{\pi}{4}\right)^3\right] + \ldots$$

Q. 3 (*b*) Calculate the Jacobian $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ of the following :

$$u = x + 2y + z$$
$$v = x + 2y + 3z$$
$$w = 2x + 3y + 5z$$

Ans.
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial x} \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix}$$
$$= 1(1) - 2(-1) + 1(-1)$$
$$= 2 + 1 - 1$$
$$= 2. \quad \textbf{Ans.}$$

Q. 3 (*c*) Discuss the maxima and minima of the function: $f(x, y) = \cos x \cos y \cos(x + y)$

Ans. Hence, $f(x, y) = \cos x \cos y \cos(x + y)$
$$= \cos y\,[-\sin x \cos(x + y) - \cos x \sin(x + yz)]$$
$$= -\cos y\,[\sin x \cos(x + y) + \cos x \sin(x + y)]$$
$$= -\cos y \sin(x + y)$$
$$[\because \sin A \cos B + \cos A \sin = \sin(A + B)]$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = -\sin(2x+y)\cos y$$

Similarly,

$$\frac{\partial f}{\partial y} = -\sin(x+2)\cos x$$

For stationary values,

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

Now, $\frac{\partial f}{\partial x} = 0$,

$\Rightarrow -\sin(2x+y)\cos y = 0$

$\Rightarrow \quad \sin(2x+y) = 0$

$\Rightarrow \quad \sin(2x+y) = \sin\pi$

$\Rightarrow \quad 2x+y = \pi - (1)$

Similarly,

$$\frac{\partial f}{\partial y} = 0$$

$\Rightarrow \quad x + 2y = \pi$ ...(2)

Solving (1) and (2),

$$x = y = \frac{\pi}{3}$$

$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is stationary point.

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = -2\cos(2x+y)\cos y$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\cos(x+2y)\cos x$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -2\cos(x+2y)\cos x$$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$,

$$r = -2\cos\pi\cos\frac{\pi}{3} = -2(-1)\left(\frac{1}{2}\right) = 1$$

$$s = -\cos\pi\cos\frac{\pi}{3} = -(-1)\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$t = -2\cos\pi\cos\frac{\pi}{3} = (-2)(-1)\left(\frac{1}{2}\right) = 1$$

Now, $\quad xt - s^2 = (1)(1) - \left(\frac{1}{2}\right)^2$

$$= \frac{3}{4}\ (+\text{ve})$$

Also, $\quad r = 1\ (+\text{ve})$

$\therefore \quad rt - s^2$ is +ve

and r is also +ve

$\therefore f$ is minimum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$\text{Minimum value} = \cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{2\pi}{3}$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{-1}{2}$$

$$= -\frac{1}{8}.$$ **Ans.**

Q.3. (*d*) Find a point on the ellipse $4x^2 + y^2 = 4$ nearest to the point (1, 2).

Ans. Given ellipse is

$$4x^2 + y^2 = 4$$

Let $P(x, y)$ be the point on given ellipse nearest to $Q(1, 2)$

$$\therefore \quad PQ = \sqrt{(x-1)^2 + (y-2)^2}$$

[By distance formula].

Let $\quad f = (x-1)^2 + (y-2)^2$

$\phi = 4x^2 + y^2 - 4$

By log range is method,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

$\Rightarrow \quad 2(x-1) + \lambda(8x) = 0$ ...(1)

$\quad 2(y-2) + \lambda(2y) = 0$ ...(2)

From (1), $\quad x - 1 + 4x\lambda = 0$

$$\Rightarrow \quad x = \frac{1}{1+4\lambda}$$

From (2), $\quad y - 2 + \lambda y = 0$

$$\Rightarrow \quad y = \frac{2}{1+\lambda}$$

Putting x and y in $4x^2 + y^2 = 4$, we get

$$\frac{4}{(1+4\lambda)^2} + \frac{4}{(1+\lambda)^2} = 4, \text{ we get}$$

Solving above, we shall obtain λ, and then x and y to obtain desired point.

Q.3. (e) Find the extreme value of $x^2+y^2+z^2$ subject to the condition $xy+yz+zx=p$.

Ans. Given function

$$f = x^2+y^2+z^2$$

condition is $xy+yz+zx=p$

$\Rightarrow \quad p = xy+yz+zx-p$

By log range's method,

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$$

$\Rightarrow \quad 2x+\lambda(y+z)=0 \quad ...(1)$

$2y+\lambda(x+z)=0 \quad ...(2)$

$2x+\lambda(x+y)=0 \quad ...(3)$

Eqn. (1) – Eqn (2) gives,

$x-y+\lambda(y-x)=0$

$x=y$

Eqn. (2) – Eqn. (3) gives,

$y-z+\lambda(z-y)=0$

$y=z$

Thus, $\quad x=y=z$

$\because \quad xy+yz+zx=p$

$\Rightarrow \quad 3x^2=p$

$\Rightarrow \quad x^2=\dfrac{p}{3}$

$\Rightarrow \quad y^2=\dfrac{p}{3}=z^2$

$\therefore$ Extreme value of

$$f=x^2+y^2+z^2$$

$$=\frac{p}{3}+\frac{p}{3}+\frac{p}{3}=p. \quad \textbf{Ans.}$$

Q.3. (f) If $f(x,y)=x^2\ y^{1/10}$, compute the value of f when $x=1.99$ and $y=3.01$.

Ans. Hence $\quad f=x^2\ y^{1/10}$

To calculate $f(1.99, 3.01)$

Let $\quad x=2,\quad y=3$

$\Rightarrow \quad x+\delta x=1.99,\quad y+\delta y=3.01$

$\Rightarrow \quad \delta x=-0.01,\quad \delta y=+0.01$

$\because \quad f=x^2\ y^{1/10}$

$$\Rightarrow \quad \delta f=\frac{\partial f}{\partial x}\delta x+\frac{\partial f}{\partial y}\delta y$$

$$\Rightarrow \quad \delta f=2xy^{1/10}\,\delta x+\frac{1}{10}\,x^2y^{-1/10}\,\delta y$$

$$\Rightarrow \quad \delta f=2(2)(3)^{1/10}.(-0.01)+\frac{1}{10}.(2)^2(3)^{-9/10}\,(0.01)$$

$$=3^{1/10}\left[-0.04+\frac{1}{10}(4)(3^{-1})(0.01)\right]$$

$$=3^{3/10}\left[-0.04+\frac{0.004}{3}\right]$$

$$=3^{-9/10}[-0.12+0.004]$$

$$=3^{-9/10}[(-0.116)]$$

$$\Rightarrow f(1.99,3.01)=f(2,3)+\delta f$$

$$=2^2(3)^{1/10}+(-0.116)3^{-9/10}$$

$$=3^{1/10}\left[4-\frac{0.116}{3}\right]$$

$$=3^{-9/10}[12-0.116]$$

$$=3^{-9/10}\times 11.884$$

$$=0.3720\times 11.884=4.4213.\quad \textbf{Ans.}$$

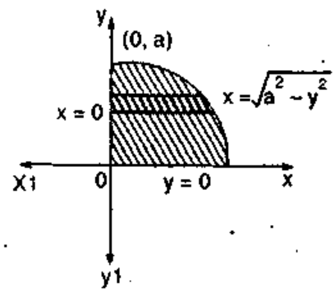
Q.4. Attempt any Four parts : (5×4=20)

(a) Evaluate the following by changing into polar co-ordinates : $\int_0^a \int_0^{\sqrt{a^2-y^2}} dydx$.

Ans. Given integral is

$$I=\int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} y^2\sqrt{(x^2+y^2)}\ dy\,dx$$

The region of integration is shown in figure.



Changing the polars.

$x=r\cos\theta,\ y=r\sin\theta$

$\Rightarrow \quad dx\,dy=rd\,rd\theta$

Also, limits are $0\le r\le a,\ 0\le\theta\le\dfrac{\pi}{2}$

$$\Rightarrow I=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}(r\sin\theta)^2\sqrt{r^2}\ rd\,rd\theta$$

$$=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^4\sin^2\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^5}{5}\right]_{r=0}^{a} \sin^2\theta \, d\theta$$

$$= \int_{0=0}^{\pi/2} \frac{a^5}{5} \sin^2\theta \, d\theta$$

$$= \frac{a^5}{5} \int_{\theta=0}^{\pi/2} \sin^2\theta \, d\theta$$

$$= \frac{a^5}{5} \int_{\theta=0}^{\pi/2} \sin^2\theta \cos a\theta \, d\theta$$

$$= \frac{a^5}{5} \left[\frac{\Gamma\frac{2+1}{2} \, \Gamma\frac{0+1}{2}}{2\Gamma\frac{2+0+2}{2}} \right]$$

$$\because \int_{0}^{\pi/2} \sin^2\theta \cos a\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}$$

$$= \frac{a^5}{5} \cdot \frac{\Gamma\frac{3}{2} \, \Gamma\frac{1}{2}}{2 . \Gamma 2} = \frac{a^5}{5} \cdot \frac{\frac{1}{2}\sqrt{\pi}\,\sqrt{\pi}}{1} = \frac{\pi a^5}{20}. \text{ Ans.}$$

Q.4. (b) Find the area enclosed between the parabola $y = 4x - x^2$ and the line $y = x$.

Ans. Given parabola is $y = 4x - x^2$

Given line is $y = x$

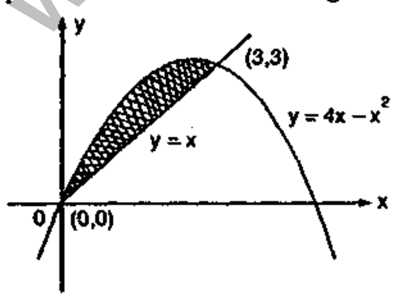
Now, $y = 4x - x^2$

$= -(x^2 + 4x)$

$= -[(x-2)^2 - 4]$

$\Rightarrow \quad y - 4 = -(x-2)^2$

Required area = area of shaded region R



$$= \iint_C dy\,dx$$

Limits are, $x \le y \le 4x - x^2$

$0 \le x \le 3.$

$$\Rightarrow \text{Required area} = \int_{x=0}^{3} \int_{y=x}^{4x-x^2} dy\,dx$$

$$= \int_{x=0}^{3} [y]_{y=x}^{4x-x^2} dx$$

$$= \int_{x=0}^{3} (4x - x^2 - x)\,dx$$

$$= \int_{0}^{3} (3x - x^2)\,dx = \left[3\frac{x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{27}{2} - 9 = \frac{9}{2}. \quad \text{Ans.}$$

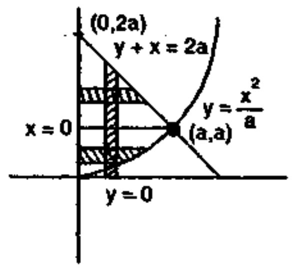
Q.4 (c) Change the order of integration in $\int_0^a \int_{x^2/a}^{2a-x} f(x,y)\,dx\,dy$.

Ans. Given integral $I = \int_{x=0}^{a} \int_{y=x^2/a}^{2a-x} f(x,y)\,dx\,dy$

Region of integration is shown in the figure by shaded portion.

By changing the order of integration

$$I = \iint_{R_1} f(x_1 y)\,dx\,dy + \iint_{R_2} f(x_2 y)\,dx\,dy$$



For R_1, $\quad 0 \le x \le \sqrt{ay}$

$0 \le y \le a$

For R_2 $\quad 0 \le x \le 2a - y$

$a \le y \le 2a$

$$\Rightarrow \quad I = \int_{y=0}^{a} \int_{y=0}^{\sqrt{ay}} f(x,y)\,dx\,dy$$

$$+\int_{x=0}^{2a}\int_{x=0}^{2a-y} f(x,y)\,dx\,dy. \text{ Ans.}$$

Q.4. (*d*) Find the volume of the solid surrounded by the surface $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1$

Ans. Given surface is

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1$$

Required volume $= 3\iiint_{V'} dx\,dy\,dz$

where V is the region,

$$x\ge 0,\quad y\ge 0,\quad z\ge 0$$

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}<1$$

Put $\left(\frac{x}{a}\right)^{2/3}=X,\ \left(\frac{y}{b}\right)^{2/3}=Y,\ \left(\frac{z}{c}\right)^{2/3}=Z,$

$$\Rightarrow\quad \frac{x}{a}=X^{3/2},\quad \frac{y}{b}=Y^{3/2},\quad \frac{z}{c}=Z^{3/2}$$

$$\Rightarrow\quad x=aX^{3/2},\quad y=bY^{3/2},\quad z=cZ^{3/2}$$

$$\Rightarrow\quad dx=\frac{3a}{2}X^{\frac{1}{2}}\,dX,$$

$$dy=\frac{3b}{2}Y^{\frac{1}{2}}\,dY,\quad dz=\frac{3c}{2}Z^{\frac{1}{2}}\,dZ$$

Required volume $=3.\iiint_{V'}\frac{3a}{2},\frac{3b}{2},\frac{3c}{2}$

$X^{\frac{1}{2}}\,Y^{\frac{1}{2}}\,Z^{\frac{1}{2}}\,dX\,dY\,dZ$

where V' is the region $X\ge 0,\ Y\ge 0,\ Z\ge 0$

$$X+Y+Z\le 1$$

$\Rightarrow$ Required volume

$$=27abc\iiint_{V'} X^{\frac{3}{2}-1}\,Y^{\frac{3}{2}-1}\,Z^{\frac{3}{2}-1}\,dX\,dY\,dZ$$

$$=27abc\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma\left(\frac{3}{2}+\frac{3}{2}+\frac{3}{2}+1\right)}$$

[By Dirchlet's Theorem)]

$$=27abc\,\frac{\frac{1}{2}\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{\Gamma\frac{11}{2}}$$

$$=\frac{27abc}{2}\,\frac{\pi}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}$$

$$=X=\frac{4abc\pi}{35}.\ \text{Ans.}$$

Q.4. (*e*) Define Gamma and Beta functions. Prove that $B(l,m)\,B(l+m,n)\,B(l+m+n,p)$

$$=\frac{\Gamma l\,\Gamma m\,\Gamma n\,\Gamma p}{\Gamma(l+m+n+p)}$$

Ans. Gamma Function : Gamma function denoted by Γn is defined as

$$\Gamma n=\int_0^\infty e^{-x}x^{n-1}\,dx\quad n>0$$

Beta Function : Beta function is denoted by $B(m, n)$ and is defined as

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

2nd Part : To prove that

$$B(l,m)\,B(l+m,n)\,B(l+m+n+p)=\frac{\Gamma l\,\Gamma m\,\Gamma n\,\Gamma p}{\Gamma(l+m+n+p)}$$

$$\text{L.H.S}=B(l,m)\,B(l+m,n).\,B(l+m+n,p)$$

$$=\frac{\Gamma e\,\Gamma m}{\Gamma(e+m)}\cdot\frac{\Gamma(e+m)\,\Gamma n}{\Gamma(l+m+n)}\cdot\frac{\Gamma(l+n+m)\,\Gamma p}{\Gamma(l+m+n+p)}$$

$$\left[\because B(p,q)=\frac{\Gamma p\,\Gamma q}{\Gamma(p+q)}\right]$$

$$=\frac{\Gamma e\,\Gamma m\,\Gamma n\,\Gamma p}{\Gamma(l+m+n+p)}=\text{R.H.S.}$$

Q.4. (*f*) Show that $\int_0^1 x^5(1-x^3)^{10}\,dx=\frac{1}{396}$

Ans. Put $\quad I=\int_0^1 x^5(1-x^3)^{10}\,dx$

Put $\quad x^3=y$

$\Rightarrow \quad x = y^{\frac{1}{3}}$

$\Rightarrow \quad dx = \frac{1}{3} y^{-\frac{2}{3}} dy$

At $\quad x = 0, \quad y = 0$

At $\quad x = 1, \quad y = 1$

$\Rightarrow \quad I = \int_0^1 y^{\frac{5}{3}} (1-y)^{10} \frac{1}{3} y^{-\frac{2}{3}} dy$

$= \frac{1}{3} \int_0^1 y (1-y)^{10} dy$

$= \frac{1}{3} \int_0^1 y^{2-1} (1-y)^{11-1} dy$

$= \frac{1}{3} B(2, 11)$

$\left[\because \int_0^1 x^{m-1}(1-x)^{n-1} dx = B(m, n)\right]$

$= \frac{1}{3} \frac{\overline{|2}\,\overline{|11}}{\overline{|2+11}} \qquad \left[\because B(m.n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}\right]$

$= \frac{1!\times 10!}{3\times 12!} \qquad [\because \overline{|n+1} =$

$= \frac{1}{3} \frac{1!\times 10!}{12\times 11\times 10!}$

$= \frac{1}{3\times 11\times 12}$

$= \frac{1}{396}$.

Q. 5. Attempt any Three parts : **(5×4=20)**

(a) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then show that

(i) $\nabla(\vec{a}.\vec{r}) = \vec{a}$, where $\vec{a}$ is a constant vector.

(ii) $grad\, r = \frac{\vec{r}}{r}$

(iii) $grad \frac{1}{r} = -\frac{\vec{r}}{r}$

(iv) $grad\, r^n = n\, r^{n-2} \vec{r}$ where $r = |\vec{r}|$

Ans. (i) $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Let $\quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\Rightarrow \quad \vec{a}.\vec{r} = a_1 x + a_2 y + a_3 z$

$\Rightarrow \nabla(\vec{a}.\vec{r}) = \hat{i} \frac{\partial}{\partial x}(a_1x + a_2y + a_3z)$

$+ \hat{j} \frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \hat{k} \frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$

$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$= \vec{a} \qquad \left[\because \text{grad}\, \phi = \sum \frac{i\partial\phi}{\partial x}\right]$

Before Proving (*ii*) and (*iii*), we shall prove (*iv*)

i.e., grad $r^n = nr^{n-2}\vec{r}$ $n\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow \quad r = |\vec{r}|$

$\Rightarrow \quad r = \sqrt{x^2 + y^2 + z^2}$

Now, grad f $= \sum i \frac{\partial\phi}{\partial x}$

$\Rightarrow \quad \text{grad}\, r^n = \sum i \frac{\partial}{\partial r}(r^n) . \frac{\partial r}{\partial x}$

$\Rightarrow \quad \text{grad}\, r^n = \sum i\, (nr^{n-1}) \frac{x}{r}$

$= \sum nr^{n-2} x\, i = nr^{n-2} \sum x\, i$

$= \sum nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$

$\Rightarrow \quad \text{grad}\, r^n = nr^{n-2}\vec{r}$ ...(2)

Hence Proved.

(*ii*) To prove grad $r = \frac{\vec{r}}{r}$

Putting $\quad n = 1$ in (2),

grad $r^1 = 1 r^{1-2}\vec{r}$

$\Rightarrow \quad \text{grad}\, r = \frac{\vec{r}}{r}$

(*iii*) To prove grad $\frac{1}{r} = -\frac{\vec{r}}{r^3}$

Putting $\quad n = -1$ in (2), we get

$$\text{grad } r^{-1} = (-1)\, r^{-1-2}\, \vec{r}$$

$$\Rightarrow \text{grad } \frac{1}{r} = -\frac{\vec{r}}{r^3}$$

Q.5. (*b*) Prove that $\vec{a} \times (\nabla \times \vec{r}) = \nabla(\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \nabla)\vec{r}$ where $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Ans. To prove that

$$\vec{a} \times (\nabla \times \vec{r}) = \nabla(\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \nabla)\vec{r}.$$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Now,

$$(\nabla \times \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0) = 0$$

$$\Rightarrow \nabla \times \vec{r} = 0$$

$$\Rightarrow \vec{a} \times (\nabla \times \vec{r}) = 0 \qquad ...(1)$$

Now

$$\vec{a} \cdot \vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\Rightarrow \vec{a} \cdot \vec{r} = a_1x + a_2y + a_3z$$

$$\Rightarrow \nabla(\vec{a} \cdot \vec{r}) = \hat{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \hat{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \hat{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$$

$$= \hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3$$

$$\Rightarrow \nabla(\vec{a} \cdot \vec{r}) = \vec{a} \qquad ....(2)$$

$$(\vec{a} \cdot \nabla) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$= a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$\Rightarrow (\vec{a} \cdot \nabla)\vec{r} = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$= \vec{a} \qquad ....(3)$$

From (2) and (3)

$$\nabla(\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \nabla)\vec{r} = \vec{a} - \vec{a}$$

$$\Rightarrow \nabla(\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \nabla)\vec{r} = 0 \qquad ...(4)$$

From (1) and (4),

$$\vec{a} \times (\nabla \cdot \vec{r}) = \nabla(\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \nabla)\vec{r}$$

Q.5. (*c*) State the Green's theorum. Verify it by evaluating $\iint\limits_C \left[\left(x^3 - xy^3\right)dx + \left(y^2 - 2xy\right)dy\right]$ where *C* is the square having the vertices at the points (0, 0), (2, 0), (2, 2) and (0, 2).

Ans. Green's Theorem : Green's theorem states that

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

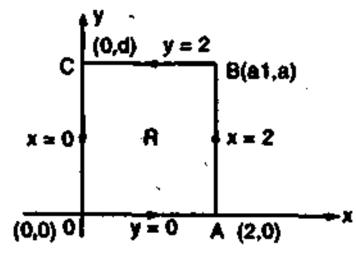
where *C* is a closed curve bounding the region *R*.

2nd Port : Given integral

$$I = \int_C [x^3 - xy^3 + dx + (y^2 - 2xy)\, du]$$

$$= \int_C [Pdx + Qdy]$$

where *C* is the square shown in figure.



Now, to verify Green's theorem we have to show,

$$\int_C (Pdx+Qdy) = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dx\, dy$$

$$+ \int_{AB} (Pdx+Qdy) + \int_{BC} (Pdx+Qdy)$$

$$+ \int_{CO} (Pdx+Qdy) \qquad \text{...(1)}$$

Along $OA: y=0 \Rightarrow dy=0$

x varies from 0 to 2

$$\therefore \int_{OA} Pdx+Qdy = \int_0^2 x^3 dx = \left[\frac{x^4}{4}\right]_0^2 = 4 \qquad \text{...(2)}$$

Along $AB: \quad x=2 \Rightarrow dx=0$

y varies from 0 to 2

$$\therefore \int_{AO} (Pdx+Qdy) \int_0^2 (y^3-4y)\, dy$$

$$= \left[\frac{y^3}{3}-2y^2\right]_0^2$$

$$= \frac{8}{3}-8 = -\frac{16}{3} \qquad \text{...(3)}$$

Along $BC, \quad y=2, \Rightarrow dy=0$

x varies from 2 to 0.

$$\therefore \int_{BC} (Pdx+Qdy) = \int_2^0 (x^3-8x)\, dx$$

$$= \left[\frac{x^4}{4}-\frac{8x^2}{2}\right]_2^0 = \left(\frac{x^4}{4}-4x^2\right)_2^0$$

$$= -\frac{16}{4}+16 = 12 \qquad \text{...(4)}$$

Along $CO, \quad x=0, dx=0$

y varies from 2 to 0

$$\therefore \int_{CO} Pdx+Qdy = \int_2^0 y^2\, dy$$

$$= \left[\frac{y^3}{3}\right]_2^0 = -\frac{8}{3} \qquad \text{...(5)}$$

Putting values from (2), (3), (4) and (5), (1) gives

$$\int_C Pdx+Qdy = 4-\frac{16}{3}+12-\frac{8}{3}$$

$$\Rightarrow \qquad \int_C Pdx+Qdy = 8 \qquad \text{...(6)}$$

Again, $\dfrac{\partial Q}{\partial x} = \dfrac{\partial}{\partial x}(y^2-2xy) = -2y$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^3-xy^3) = -3xy^2$$

$$\therefore \qquad \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \iint_R (-2y+3xy^2)\, dx\, dy$$

Limits are, $0 \le x \le 2$

$0 \le y \le 2$

$$\text{d}\Rightarrow \quad \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \int_{x=0}^2 \int_{y=0}^2 (-2y+3xy^2)\, dx\, dy$$

$$= \int_{x=0}^2 \left[-y^2+3x\frac{y^3}{3}\right]_{y=0}^2 dx$$

$$= \int_{x=0}^2 \left(-4x+4x^2\right)_0^2$$

$$= \left(-4x+4x^2\right)_0^2 = -8+16 = 8 \qquad \text{...(7)}$$

From (6) and (7)

$$\int_C Pdx+Qdy = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dx\, dy$$

Hence Green's theorem is verified.