B. TECH.

EAC-103

FIRST SEMESTER EXAMINATION, 2008-09

MATHEMATICS–I

Time : 3 Hours *Total Marks :* **100**

*Note : (i) Attempt **all** questions.*
*(ii) All questions carry **equal** marks.*
(iii) Be precise in your answer.
(iv) No second answer book will be provided.

SECTION—A

Q. 1. All parts of this question are compulsory.
$2 \times 10 = 20$

1. (*a*) For which value of '*b*' the rank of the matrix

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix} \text{ is 2, } b =$$

Ans. Justification :

$\therefore$ Rank of A = 2 < order of matrix

$$\Rightarrow \begin{vmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{vmatrix} = 0$$

$\Rightarrow 1(30-26) - 5(-2b) + 4(-3b) = 0$

$\Rightarrow 4 + 10b - 12b = 0$

$\Rightarrow b = 2$

1. (*b*) Determine the constants *a* and *b* such that the curl of vector $\overline{A} = (2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}$ is zero, *a* = , *b* =

Ans. Justification :

$\therefore \nabla \times \vec{A} = 0$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+3yz & x^2+axz-4z^2 & -3xy-byz \end{vmatrix} = 0$$

$$\Rightarrow \hat{i}[-3x - b^2 - 9x + 8z] - \hat{j}[-3y + 3y] + \hat{k}[2x + 9z - 2x - 3z] = 0$$

$$\Rightarrow \hat{i}[(-3-a)x + (B-b)z] + \hat{k}(a-3)z = 0$$

$$\Rightarrow a = -3, b = 8$$

1. (*c*) The n^{th} derivative (y_n) of the function $y = x^2 \sin x$ at $x = 0$ is

Ans. Justification: $y = x^2 \sin x$

$$\Rightarrow y_n = x^2 \frac{d^n}{dx^n}(\sin x) + {}^nC_1(2x)\frac{n^{x-1}}{dx^{n-1}}(\sin x) + {}^nC_2(2)\frac{d^{n-2}}{dx^{n-2}}(\sin x)$$

$$y_n(x) = x^2 \sin\left(x + \frac{nx}{2}\right) + n(2x)\sin\left[x + (n-1)\frac{\pi}{2}\right] + n(n-1)\sin\left[x + (n-2)\frac{\pi}{2}\right]$$

$$\Rightarrow y_n = n(n-1)\sin(n-2)\frac{\pi}{2}$$

$$= -n(n-1)\sin\frac{n\pi}{2}$$

$$= (n-n^2)\sin\frac{n\pi}{2}$$

Q. 1. (*d*) With usual notations, match the items on right hand side with those on left hand side for properties of maximum and minimum.

(*i*) Maxim. (*s*)	(*p*) $rt-s^2=0$
(*ii*) Minm. (*r*)	(*q*) $rt-s^2<0$
(*iii*) Saddle point (*q*)	(*r*) $rt-s^2>0$ and $r>0$
(*iv*) Failure case (*p*)	(*s*) $rt-s^2>$ and $r>0$

Ans.

(*i*) Max^m (*s*)	(*s*) $rt-s^2>0$ and $r>0$
(*ii*) Min^m	(*r*) $rt-s^2>0$ and $r>0$
(*iii*) Saddle point	(*q*) $rt-s^2<0$
(*iv*) Failure case	(*p*) $rt-s^2=0$

1. (*e*) Match the items on the right hand side with those on left hand side for the following special functions: (Full marks is awarded if all matchings are correct).

(*i*) $\beta(p,q)$	(*p*) $\Gamma(1/2)$
(*ii*) $\frac{\Gamma p\,\Gamma q}{\Gamma p+q}$	(*q*) $\int_0^\infty \frac{y^{p-1}}{(1+y)(p+q)}dy$
(*iii*) $\sqrt{\pi}$	(*r*) $\beta(P,q)$
(*iv*) $\frac{\pi}{\sin p\pi}$	(*s*) $\Gamma p\,\Gamma 1-p$

Ans.

(*i*) $\beta(p,q)$	(*p*) $\beta(P,q)$
(*ii*) $\frac{\Gamma p\,\Gamma q}{\Gamma p+q}$	(*q*) $\int_0^\infty \frac{y^{p-1}}{(1+y)^{(p+q)}}dy$
(*iii*) $\sqrt{\pi}$	(*r*) $\beta(p,q)$
(*iv*) $\frac{\pi}{\sin p\pi}$	(*s*) $\Gamma p\,\Gamma 1-p$

Indicate True or False for the following statements:

Q. 1. (*f*) (*i*) If $|A|=0$, then at least one eigen value is zero. (True/ False)

(*ii*) A^{-1} exists iff 0 is an eigen value of A. (True/ False)

(*iii*) If $|A| \neq 0$, then A is known as singular matrix. (True/ False)

(*iv*) Two vectors X and Y is said to be orthogonal Y, $X^TY = Y^TX \neq 0$. (True/ False)

Ans. (*i*) True

(*ii*) False

(*iii*) False

(*iv*) False

Q. 1. (*g*) (*i*) The curve $y^2 = 4ax$ is symmetric about *x*-axis. (True/ False)

(*ii*) The curve $x^3 + y^3 = 3axy$ is symmetric about the line $y = -x$. (True/ False)

(*iii*) The curve $x^2 + y^2 = a^2$ is symmetric about both the axis *x* and *y*. (True/ False)

(*iv*) The curve $x^3 - y^3 = 3axy$ is symmetric about the line $y = x$. (True/ False)

Ans. (*i*) True

(*ii*) False

(*iii*) True

(*iv*) False

Pick the correct answer of the choices given below:

Q. 1. (*h*) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is position vector, then value of $\nabla(\log r)$ is

(*i*) $\frac{\vec{r}}{r}$ (*ii*) $\frac{\vec{r}}{r^2}$

(*iii*) $-\frac{\vec{r}}{r^3}$ (*iv*) none of the above

Ans. (*ii*) $\frac{\vec{r}}{r^2}$

Justification :

$$\nabla(\log r) = \Sigma\,\hat{i}\,\frac{\partial}{\partial x}(\log r)$$

$$= \Sigma\,\hat{i}\,\frac{\partial}{\partial r}(\log r)\frac{\partial r}{\partial x}$$

$$= \Sigma \hat{i} \frac{1}{r} \cdot \frac{x}{r} \left[\because \frac{\partial r}{\partial x} = \frac{x}{r} \right]$$

$$= \frac{15}{r^2} x\hat{i}$$

$$= \frac{\vec{r}}{r^2}$$

1. (i) The Jacobian $\frac{\partial(uv)}{\partial(xy)}$ for the function $u = e^x \sin y$, $v = (x + \log \sin y)$ is

(i) 1

(ii) $\sin x \sin y - xy \cos x \cos y$

(iii) 0

(iv) $\frac{e^x}{x}$

Ans. (iii) 0

Justification: $\partial u = e^x \sin y,\ y = x + \log \sin y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} e^x \sin y & e^x \cos y \\ 1 & \frac{1}{\sin y} \cos y \end{vmatrix} = 0$$

1. (j) The volume of the solid under the surface $az = x^2 + y^2$ and whose base R is the circle $x^2 + y^2 = a^2$ is given as

(i) $\pi | 2a$

(ii) $\pi a^3 | 2$

(iii) $\frac{4}{3} \pi a^3$

(iv) None of the above

Ans. (ii) $\frac{\pi a^3}{2}$

Justification : Required Volume $= \iiint dx\, dy\, dz$

$$= \iint_R \int_{z=0}^{\frac{x^2+y^2}{a}} dx\, dy\, dz$$

$$= \iint_R \frac{x^2 + y^2}{a} dx\, dy$$

$$= \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \frac{r^2 r\, dr\, d\theta}{a} \quad \begin{bmatrix} \text{Put } x = r\cos\theta \\ y = r\sin\theta \end{bmatrix}$$

$$= \frac{\pi a^3}{2}$$

SECTION-B

Q. 2. Attempt any three parts of the following: **$10 \times 3 = 30$**

2. (a) If $y = \left(\sin^{-1} x\right)^2$, prove that $y_n(0) = 0$ for b odd and $y_n(0) = 2, 2^2, 4^2, 6^2, \ldots, (n-2)^2$, $n \neq 2$ for n is even.

Ans. $\therefore \quad y = (\sin^{-1} x)^2 \qquad \ldots(1)$

$$\Rightarrow y_1 = 2(\sin x) \cdot \frac{d}{dx}(\sin^{-1} x)$$

$$\Rightarrow y_1 = 2(\sin^{-1} x) \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\, y_1 = 2\sin^{-1} x$$

$$\Rightarrow (1-x^2) y_1^2 = 3(\sin^{-1} x)^2$$

$$\Rightarrow (1-x^2)\, y_1^2 = 4y \quad \ldots(2) \left[\therefore y = (\sin^{-1} x)^2 \right]$$

Differentiating (2), we get

$$\frac{d}{dx}\left[(1-x^2) y_1^2\right] = \frac{d}{dx}(4y)$$

$$\Rightarrow -2xy_1^2 + (1-x^2)\frac{d}{dx}(y_1^2) = 4y_1$$

$$\Rightarrow -2xy_1^2 + (1-x^2) 2y_1 y_2 = 4y_1$$

$$(1-x^2)y_2 - xy - 2 = 0 \quad ...(3)$$

Differentiating (3) n times by

$$[(1-x^2)y_{n+2} + {}^nC_1(-2x)\, y_{n+1} + {}^nC_2(-2)y_n]$$

$$-[xy_{n+1} + {}^nC_1(1)y_n] - 0 = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}$$

$$-xy_{n+1} - ny_n = 0$$

$$\left[\therefore {}^nC_1 = n, {}^nC_2 = \frac{n(n-1)}{2}\right]$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0 \quad ...(4)$$

From (1), $y(0) = [\sin^{-1}(0)]^2 = 0$

From (2), $y_1(0) = 0$

Putting $x = 0$ in (3), $y_2(0) = 2$

Putting $x = 0$ in (4),

$$y_{n+2}(0) = n^2y_n(0) = 0$$

$$\Rightarrow y_{n+2}(0) = n^2 y_n(0) \quad ...(5)$$

Putting $n = 1$ in (5), $y_3(1) = 1^2 y_1(0) = 0$

Putting $n = 3$ in (5), $y_5(0) = 3^2 y_3(0) = 0$

Putting $n = 5$ in (5), $y_7(0) = 5^2 y_5(0) = 0$

$$\Rightarrow y_n(0) = 0 \text{ if } n \text{ is odd.}$$

Again, putting $n = 2$ in (5), $y_4(0) = 2^2 y_2(0) = 2^2$

Putting $n = 4$ in (5), $y_6(0) = 4^2 y_4(0) = 2.2^2.4^2$

Putting $n = 6$ in (5), $y_8(0) = 6^2 y_6(0) = 2.2^2.4^2.6^2$

: : :

Putting $n = n-2$ in (5),

$$y_n(0) = \ ... = 2.2^2.4^2.6^2 - (n-2)^2 \text{ if } n \text{ is even}$$

Thus,

$$y_n(0) = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 2.2^2.4^2.6^2...(n-2)^2 \text{ if } n \text{ is even} \end{cases}$$

Q. 2. (*b*) Find the dimension of rectangular box of maiximum capacity whose surfae area is given when (*a*) box is open at the top (*b*) box is closed.

Ans. Let x, y and z be the length, breadth and height of the box. Let S & Y be surface area & volume of box respectively.

(*a*) When box is open at the top.

Then

$$S = 2(xy + yz + zx) - xy$$

$$S = xy + 2(x+y)z \quad ...(1)$$

and $\quad Y = xyz \quad ...(2)$

By Lagrange's method,

$$\frac{\partial Y}{\partial x} + \lambda\frac{\partial S}{\partial x} = 0, \frac{\partial Y}{\partial y} + \lambda\frac{\partial S}{\partial y} = 0, \frac{\partial Y}{\partial z} + \lambda\frac{\partial S}{\partial z} = 0$$

$$\Rightarrow \quad yz + \lambda(y + 2z) = 0 \quad ...(3)$$

$$xz + \lambda(x + 2z) = 0 \quad ...(4)$$

$$xy + 2\lambda(x+y) = 0 \quad ...(5)$$

Eq^n(3) – Eq^n(4) gives

$$(y-x)z + \lambda(y-x) = 0$$

$$\Rightarrow \quad y = x.$$

$$x(2z-y) + 2x(2z-y) = 0$$

$$\Rightarrow \quad y = 2z$$

$\Rightarrow$ For surface area (S) given , the dimensions for maximum capacity are $x = y = 2z$.

Now

$$S = xy + 2(x+y)z \quad \text{[From (1)]}$$

$$= (2z)(2z) + 2(2z + 2z)z$$

$$= 4z^2 + 8z^2$$

$$= 12\, z^2$$

$$\Rightarrow \quad 8z = \frac{\sqrt{5}}{\sqrt{12}}$$

$$\Rightarrow \quad z = \frac{\sqrt{5}}{2\sqrt{3}}$$

$\therefore$ Length $x = 2z = \dfrac{2\sqrt{5}}{2\sqrt{3}} = \dfrac{\sqrt{6}}{\sqrt{3}}$

Breadth $y = 2z = \dfrac{\sqrt{5}}{\sqrt{3}}$

Height $z = \dfrac{\sqrt{5}}{2\sqrt{3}}$

(*b*) When box is closed,

then

$$S = 2(xy + yz + zx) \quad \text{...(1)}$$

and $$Y = xyz \quad \text{...(2)}$$

$\therefore$ By Lagrange's method,

$$\frac{\partial S}{\partial x} + \lambda \frac{\partial V}{\partial x} = 0$$

$$\frac{\partial S}{\partial y} + \lambda \frac{\partial V}{\partial y} = 0$$

$$\frac{\partial S}{\partial z} + \lambda \frac{\partial V}{\partial z} = 0$$

$$\Rightarrow \quad 2(y+z) + \lambda yz = 0 \quad \text{...(3)}$$

$$2(x+z) + \lambda xz = 0 \quad \text{...(4)}$$

$$2(x+y) + \lambda xy = 0 \quad \text{...(5)}$$

Eq^n(3) – Eq^n(4) gives

$$2(y-x) + \lambda z(y-x) = 0$$

$$\Rightarrow \quad y = x.$$

Eq^n(4) – Eq^n(5) gives

$$2(z-y) + \lambda x(z-y) = 0$$

$$\Rightarrow \quad y = z$$

Thus, $x = y = z$.

$$\therefore \quad S = 2(xy + yz + zx)$$

$$\Rightarrow \quad S = 2(z.z + z.z + z.z) \quad [\because x = y = z]$$

$$\Rightarrow \quad S = 6z^2$$

$$\Rightarrow \quad z = \sqrt{\frac{5}{6}}$$

$\therefore$ height of box $x = \sqrt{\dfrac{5}{6}}$

Breadth of box $y = \sqrt{\dfrac{5}{6}}$

Height of box $z = \sqrt{\dfrac{5}{6}}$

Q. 2. (*c*) Find a matrix P which diagonalizes the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, verify $P^{-1}AP = D$ where D is the diagonal matrix.

Ans. Given matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

Characteristic eqn of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (4-\lambda)(3-\lambda) - 2 = 0$$

$$\Rightarrow 12 - 4\lambda - 3\lambda + \lambda^2 - 2 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 10 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 2\lambda + 10 = 0$$

$$\Rightarrow \lambda(\lambda - 5) - 2(\lambda - 5) = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 5) = 0$$

$\Rightarrow \lambda = 2, \lambda = 5,$ which are eigen values of matrix A.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigen vector of A for eigen value λ.

$$\Rightarrow \quad (A - 1I)X = 0$$

$$\Rightarrow \quad \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{...(1)}$$

For $\lambda = 2$, (1) gives

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2x_1 + x_2 = 0$

or $\quad 2x_1 + x_2 = 0$

Let $x_1 = k \Rightarrow x_2 = -2k$

$\therefore$ Eigen vector for $\lambda = 2$ is $\begin{bmatrix} k \\ -2k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Again, for $\lambda = 5$ (1) gives

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -1x_1 + x_2 = 0$

and $\quad 2x_1 - 2x_2 = 0$

or $\quad x_1 - x_2 = 0$

19 Let $x_1 = k \Rightarrow x_2 = k$

$\therefore$ Eigen vector for $\lambda = 5$ is $\begin{bmatrix} k \\ k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ Modal matrix $P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$, which will diagonalize A.

Verification :

We shall calculate $P^{-1}AP$ and it should come as diagonal matrix (D).

Now $\quad P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$\Rightarrow \quad \text{adj } P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

$|P| = (1)(1) + (2)(1)$

$= 3$

$\therefore \quad P^{-1} = \dfrac{\text{adj } P}{|P|} = \dfrac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

$$\Rightarrow \quad P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4-2 & 4+1 \\ 2-6 & 2+3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -4 & 5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2+4 & 5-5 \\ 4-4 & 10+5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 6 & 0 \\ 0 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Hence verified that $P^{-1}AP$ is a diagonal matrix.

2. (*d*) Find the area and the mass contained in the first quadrant enclosed by the curve $\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} = 1$ where $\alpha > 0, \beta > 0$ given that density at any point $P(xy)$ is $k\sqrt{xy}$.

Ans. Given curve is $\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} = 1.$

and Density $p(x, y) = k\sqrt{xy}.$

Required area $A = \iint_R dx\, dy$

where R is the region given by

$x \geq 0, y \geq 0, \left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} \leq 1.$

Put $\left(\frac{x}{a}\right)^{\alpha} = X \qquad \left(\frac{y}{b}\right)^{\beta} = Y$

$\Rightarrow \frac{x}{a} = X^{\frac{1}{\alpha}} \qquad \frac{y}{b} = Y^{\frac{1}{\beta}}$

$\Rightarrow x = aX^{\frac{1}{\alpha}} \qquad y = bY^{\frac{1}{\beta}}$

$$\Rightarrow dx = \frac{a}{\alpha} X^{\frac{1}{\alpha}-1} dX \qquad dy = \frac{b}{\beta} y^{\frac{1}{\beta}-1} dy$$

$$\therefore \text{Required area A} = \iint_{R^1} \frac{a}{\alpha} X^{\frac{1}{\alpha}-1} \frac{b}{\beta} Y^{\frac{1}{\beta}-1} dx\, dy$$

whre R^1 is the region $x > 0, y > 0, x + y \leq 1$.

$$\Rightarrow \text{Required area} = \frac{ab}{\alpha\beta} \iint_{R^1} X^{\frac{1}{\alpha}-1} Y^{\frac{1}{\beta}-1} dx\, dy$$

$$= \frac{ab}{\alpha\beta} \frac{\sqrt{\frac{1}{\alpha}}\sqrt{\frac{1}{\beta}}}{\sqrt{\frac{1}{\alpha}+\frac{1}{\beta}+1}}$$

[By Dirichlet's integral]

2nd Part :

$$= \iint_R k\sqrt{xy}\, dx\, dy$$

where R is the region

$$x \geq 0, y \geq 0, \left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} \leq 1.$$

Put $\left(\frac{x}{a}\right)^{\alpha} = X, \quad \left(\frac{y}{b}\right)^{\beta} = Y$

$$\Rightarrow x = aX^{\frac{1}{\alpha}} \qquad y = bY^{\frac{1}{\beta}}$$

$$\Rightarrow dx = \frac{a}{\alpha} X^{\frac{1}{\alpha}-1} dx, \; dy = \frac{b}{\beta} Y^{\frac{1}{\beta}-1} dy$$

$\therefore$ Required mass

$$M = \iint aX^{\frac{1}{\alpha}} bY^{\frac{1}{\beta}} \frac{a}{\alpha} X^{\frac{1}{\alpha}-1} \frac{b}{\beta} Y^{\frac{1}{\beta}-1} dx\, dy$$

where R^1 is the region $x > 0, y > 0, x + y \leq 1$.

$\Rightarrow$ Required mass

$$M = k\iint_{R^1} \frac{a^{\frac{1}{2}+1} b^{\frac{1}{2}+1}}{\alpha\beta} X^{\frac{1}{2\alpha}+\frac{1}{\alpha}-1} Y^{\frac{1}{2\beta}+\frac{1}{\beta}-1} dx\, dy$$

$$= \frac{ka^{\frac{3}{2}} b^{\frac{3}{2}}}{\alpha\beta} \iint_{R^1} X^{\frac{3}{2\alpha}-1} Y^{\frac{3}{2\beta}-1} dx\, dy$$

$$= \frac{ka^{\frac{3}{2}} b^{\frac{3}{2}}}{\alpha\beta} \frac{\sqrt{\frac{3}{2\alpha}}\sqrt{\frac{3}{2\beta}}}{\sqrt{\frac{3}{2\alpha}+\frac{3}{2\beta}+1}}$$ [By Dirichlet's theorm]

2. (e) Using the divergence theorem, evaluate the surface integral

$$\iint_S (yz\, dy\, dz + zx\, dz\, dx + xy\, dy\, dx)$$

where $S : x^2 + y^2 + z^2 = 4$

Ans. Given surface integral

$$I = \iint_S (yz\, dy\, dz + zx\, dx\, dz + xy\, dy\, dx)$$

$$= \iint_S \vec{F}.\hat{n}\, ds$$

where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

By divergence theorem,

$$\iint_S \vec{F}.\hat{n}\, ds = \iint_V \nabla.\vec{F}\, dy$$

Now,

$$\nabla.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)$$

$$= 0$$

$$d\iint_S \vec{F}.\hat{n}\, ds = 0$$

$$\Rightarrow \iint_S (yz\, dy\, dz + zx\, dx\, dz + xy\, dy\, dx) = 0.$$

SECTION-C

Q. 3. Attempt any two parts from each question. All questions are compulsory.

3. (*a*) Trace the curve $r^2 = a^2 \cos 2\theta$

Ans. Please see Q.3 (*d*) of 2000-2001.

3. (*b*) If $u = \log\left(\frac{(x^2+y^2)}{(x+y)}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$ **(Unit-II)**

Ans. Here $u = \log\frac{x^2+y^2}{x+y}$

$\Rightarrow \quad e^u = \frac{x^2+y^2}{x+y}$

Let $\quad V(x, y) = \frac{x^2+y^2}{x+y}$

$\Rightarrow \quad V(tx, ty) = \frac{t^2(x^2+y^2)}{t(x+y)}$

$= t\left(\frac{x^2+y^2}{x+y}\right)$

$= t\,V(x, y)$

$\therefore$ V(*x*, *y*) is a function of degree *n* = 1.

$\therefore$ By Euler's theorem,

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = \lambda V$$

$$\Rightarrow \quad x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = V$$

$$\Rightarrow \quad x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = e^u$$

$$\left[\because Y = e^u = \frac{x^2+y^2}{x+y}\right]$$

$$\Rightarrow \quad x\frac{\partial}{\partial u}(e^u)\frac{\partial u}{\partial x} + y\frac{\partial}{\partial 4}(e^u)\frac{\partial u}{\partial y} = e^u$$

$$\Rightarrow \quad xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = e^4$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

Hence proved.

3. (*c*) If $V = f(2x-3y, 3y-4z, 4z-2x)$, compute the value of $6V_x + 4V_y + 3V_z$.

Ans. Here $V = f(2x-3y, 3y-4z, 4z-2x)$

Let $2x-3y = r, 3y-4z = s, 4z-2x = t$

$\therefore \quad V = f(r,s,t)$

Now,

$$V_x = \frac{\partial V}{\partial x}$$

$$= \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial V}{\partial S}\cdot\frac{\partial s}{\partial x} + \frac{\partial V}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial V}{\partial r}\cdot(2) + \frac{\partial V}{\partial s}(0) + \frac{\partial V}{\partial t}(-2)$$

$$\Rightarrow \quad 6\times x = 12\frac{\partial V}{\partial r} - 12\frac{\partial V}{\partial t} \qquad ...(1)$$

Also,
$$V_y = \frac{\partial V}{\partial y}$$

$$= \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial V}{\partial s}\cdot\frac{\partial S}{\partial y} + \frac{\partial V}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial V}{\partial r}\cdot(-3) + \frac{\partial V}{\partial s}(3) + \frac{\partial V}{\partial t}(0)$$

$$\Rightarrow \quad 4V_y = -12\frac{\partial V}{\partial r} + 12\frac{\partial V}{\partial s} \qquad ...(2)$$

Again,
$$Y_z = \frac{\partial V}{\partial z}$$

$$= \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial V}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial V}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial V}{\partial r}.(0) + \frac{\partial V}{\partial s}(-4) + \frac{\partial V}{\partial t}(4)$$

$$\Rightarrow \quad 3y_z = -12\frac{\partial V}{\partial s} + 12\frac{\partial V}{\partial t} \quad ...(3)$$

Adding (1), (2) and (3), we get

$6V_x + 4V_y + 3Y_z = 0.$

Q. 4. (a) The temperature 'T' at any point (xyz) in space is T(xyz) = K xyz² where K is constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$

Ans. Here temperature $T = kxyz^2$.

Given sphere is $x^2 + y^2 + z^2 = a^2 \quad ...(1)$

$\Rightarrow x^2 + y^2 + z^2 - a^2 = 0.$

Let $\phi = x^2 + y^2 + z^2 - a^2$

We want to find the highest value of T.

$\therefore$ By Lagrange's method,

$$\frac{\partial T}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial T}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial T}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$

$$\Rightarrow \quad kyz^2 + 2\lambda x = 0 \quad ...(2)$$

$$kxz^2 + 2\lambda y = 0 \quad ...(3)$$

$$2kxy^2 + 2\lambda z = 0 \quad ...(4)$$

From (2), $-2\lambda = \frac{kyz^2}{x} \quad ...(5)$

From (3), $-2\lambda = \frac{kxz^2}{y} \quad ...(6)$

From (4), $-2\lambda = \frac{2kyz}{z} \quad ...(7)$

From (5) and (6),

$$\frac{kyz^2}{x} = \frac{kxz^2}{y}$$

$$\Rightarrow \quad y^2 = x^2$$

Also from (6) and (7)

$$\Rightarrow \quad z^2 = 2y^2$$

$$\Rightarrow \quad z^2 = 2x^2 \quad [\because y^2 = x^2]$$

Now, from,

$$x^2 + y^2 + z^2 = a^2$$

$$\Rightarrow \quad x^2 + x^2 + 2x^2 = a^2$$

$$\Rightarrow \quad x^2 = \frac{a^2}{4}$$

$$\Rightarrow \quad x = \pm\frac{a}{2}$$

$$\therefore \quad y^2 = x^2$$

$$\Rightarrow \quad y = \pm\frac{a}{2}$$

Also, $z^2 = 2x^2$

$$\Rightarrow \quad z^2 = \frac{a^2}{2}$$

$$\Rightarrow \quad z = \pm\frac{a}{\sqrt{2}}$$

$$T = kxyz^2$$

$$\Rightarrow \quad T_{max} = k\left(\pm\frac{a}{z}\right)\left(\pm\frac{a}{z}\right)\left(\pm\frac{a}{\sqrt{2}}\right)^2$$

$$\Rightarrow \quad T_{max} = \frac{ka^4}{4}$$

Q. 4. (b) Verify the chain rule for Jacobians if $x = u,\ y = u \tan v, z = w$

Ans. We shall verify

$$\Rightarrow \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$$

Here $x = u, y = u, \tan v, z = w$

$$\Rightarrow \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \tan u & 4\sec^2 u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 4\sec^2 u$$

To calculate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$, we shall express u, v and w in terms of x, y and z.

$\therefore x = u \Rightarrow u = x$

Also, $\quad y = u \tan v$

$$\Rightarrow \quad \tan v = \frac{y}{u}$$

$$= \frac{y}{x}$$

$$\Rightarrow \quad v = \tan^{-1}\frac{y}{x}$$

Again,

$$z = w$$

$$\Rightarrow \quad w = z$$

Thus, we have obtained

$u = x,\ v = \tan^{-1}\frac{y}{x},\ w = z.$

$$\text{Now,} \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \left(1+\frac{y^2}{x^2}\right)\left(\frac{-y}{x^2}\right) & \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{x}{x^2+y^2}$$

$$= \frac{u}{u^2+u^2\tan^2 v}$$

$$= \frac{u}{u^2(1+\tan^2 v)}$$

$$= \frac{1}{4\sec^2 v}$$

$$\Rightarrow \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

$$= 4\sec^2 Y \times \frac{1}{4\sec^2 Y}$$

$$\Rightarrow \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$$

4. (c) The time 'T' of a complete oscillation of a simple pendulum of length 'L' is governed by the equation $T = 2\pi\sqrt{\frac{L}{g}}$, g is constant. Find the approximate error in the calculated value of T corresponding to an error of 2% in the value of L.

Ans. Here $T = 2\pi\sqrt{\frac{L}{g}}$

$\therefore$ % error in L = 2%

$$\Rightarrow \frac{8L}{L} \times 100 = 2$$

$$\Rightarrow \frac{8L}{L} = \frac{2}{100}$$

$\therefore$ g is constant

$$\Rightarrow \frac{\delta g}{g} = 0.$$

Now, $T = 2\pi\sqrt{\frac{L}{g}}$

$$\Rightarrow \log T = \log 2\pi + \frac{1}{2}(\log L - \log g)$$

$$\Rightarrow \frac{\delta T}{T} = 0 + \frac{1}{2}\left(\frac{\delta L}{L} - \frac{\delta g}{g}\right)$$

$$\Rightarrow \frac{\delta T}{T} = \frac{1}{2} \times \frac{2}{100}$$

$$\Rightarrow \frac{\delta T}{T} = \frac{1}{100}$$

$\therefore$ % error in T = 1 %.

Q. 5. (*a*) Determine '*b*' such that the system of homogeneous equation $2x + y + 2z = 0$; $x + y + 3z = 0$; $4x + 3y + bz = 0$ has (*i*) Trivial solution (*ii*) Non-trivial solution. Find the Non-trivial solution using matrix method.

Ans. Given system of equation is

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + bz = 0$$

Coefficient matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & b \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & b \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & b \end{bmatrix} R_3 \to R_3 - R_2$$

(*i*) For trivial solution,

Rank A = No. of unknowns = 2

It is possible if $b - 8 \neq 0$

$\Rightarrow$ $b \neq 8$

i.e. For trivial solution $b \neq 8$.

(*ii*) For non-trivial soln.

Rank A < No. of unknowns = 2

It is possible if $b - 8 = 0$

$\Rightarrow$ $b = 8$

i.e., for non trivial soln., $b = 8$.

To obtain this soln, put $b = 8$ in A, we get

$$A \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Rewriting the system,

$$x + y + 3z = 0$$
$$-y - 4z = 0$$

Let $z = k$

$\Rightarrow$ $y = -4k$

$\therefore$ $x + y + 3z = 0$

$\Rightarrow$ $x - 4k + 3k = 0$

$\Rightarrow$ $x = k$

$\therefore$ Desired non-trivial soln. is $\left.\begin{matrix} x = k \\ y = -4k \\ z = k \end{matrix}\right\}$

Q. 5. (*b*) Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ and hence find A^{-1}.

Ans. Given matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Characteristic eq^n of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(-1-\lambda)-4 = 0$$

$$\Rightarrow \quad -1-\lambda+\lambda+\lambda^2-4 = 0$$

$$\Rightarrow \quad \lambda^2-5 = 0$$

To verify Cayley Hamilton theorem, we have to show

$$A^2 - 5I = 0$$

Now,

$$A^2 = AA$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & 2-2 \\ 2-5 & 4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Now,

$$A^2 - I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0$$

$$\Rightarrow \quad A^2 - 5I = 0$$

Thus Cayley Hamilton theorem is veified.

To find A^{-1}, we multiply above relation by A^{-1}.

We get $A^{-1}(A^2 - 5I) = 0A^{-1}$

$$\Rightarrow \quad A - 5A^{-1} = 0 \qquad [\because IA^{-1} = A^{-1}]$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}A$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Q. 5. (*c*) Find the eigen value and corresponding eigen vectors of the matrix

$$I = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$$

Ans. Given matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Characteristic eqn. of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-5-\lambda)(-2-\lambda)-4 = 0$$

$$\Rightarrow \quad (5+\lambda)(2+\lambda)-4 = 0$$

$$\Rightarrow \quad 10+5\lambda+2\lambda+\lambda^2-4 = 0$$

$$\Rightarrow \quad \lambda^2+7\lambda+6 = 0$$

$$\Rightarrow \quad \lambda^2+6\lambda+\lambda+6 = 0$$

$$\Rightarrow \quad \lambda(\lambda+6)+1(\lambda+6) = 0$$

$$\Rightarrow \quad (\lambda+1)(\lambda+6) = 0$$

$$\Rightarrow \quad \lambda = -1, -6.$$

which are eigen values.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be given vector of A corresponding to eigen value λ.

$$\Rightarrow \quad (A - \lambda I)X = 0$$

$$\Rightarrow \quad \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = -1$, (1) gives

$$\begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$\Rightarrow \quad -4x_1 + 2x_2 = 0$

or $\quad 2x_1 - x_2 = 0$

& $\quad 2x_1 - x_2 = 0$

Let $x_1 = k, \Rightarrow x_2 = 2k$

$\therefore \quad X = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}k\\ 2k\end{bmatrix}$

$k = \begin{bmatrix}1\\ 2\end{bmatrix}$ or $\begin{bmatrix}1\\ 2\end{bmatrix}$

For $\lambda = -6$, (1) gives

$$\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$\Rightarrow \quad 1x_1 + 2x_2 = 0$

and $\quad 2x_1 + 4x_2 = 0$

or $\quad x_1 + 2x_2 = 0$

Let $x_2 = k, \Rightarrow x_1 = -2k$

$\therefore \quad X = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-2k\\ +2k\end{bmatrix}$

$k = \begin{bmatrix}-2\\ 1\end{bmatrix}$ or $\begin{bmatrix}-2\\ 1\end{bmatrix}$

Thus eigen value of given matrix are -1 and -6.

Corresponding eigen vectors are $\begin{bmatrix}1\\ 2\end{bmatrix}$ and $\begin{bmatrix}-2\\ 1\end{bmatrix}$ respectively.

Q. 6. (*a*) Find the directional derivative of $\nabla(\nabla f)$ at the point (1, –2, 1) in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $f = 2x^3y^2z^4$.

Ans. To find directional dervatve of $\nabla - \nabla f$

where $\quad f = 2x^3y^2z^4.$

$$\nabla.\nabla f = \nabla^2 f$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Now.

$$\frac{\partial f}{\partial x} = 6x^2y^2z^4$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = 12xy^2z^4$$

Also $\quad \frac{\partial f}{\partial y} = 4x^3yz^4$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial y^2} = 4x^3z^4$$

and $\quad \frac{\partial f}{\partial z} = 24x^3y^2z^2$

$\Rightarrow \quad \nabla.\nabla f = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

Let $\quad \phi = \nabla.\nabla f$

$\Rightarrow \quad \phi = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

$\therefore$ Given surface is

$$xy^2z = 3x + z^2$$

$\Rightarrow \quad xy^2z - 3x - z^2 = 0$

We shall find normal $\vec{n}$ to this surface.

Let $\quad g = xy^2z - 3x - z^2$

$$\Rightarrow \quad \nabla g = \hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y} + \hat{k}\frac{\partial g}{\partial z}$$

$$\nabla g = \hat{i}[y^2z - 3] + \hat{j}[2xyz] + \hat{k}[xy^2 - 2z]$$

at given point (1, –2, 1)

$$(\nabla g)_{(1,-2,1)} = \hat{i} - 4\hat{j} + 2\hat{k}$$

$\Rightarrow \quad \vec{n} = \hat{i} - 4\hat{j} + 2\hat{k}$

$\Rightarrow \quad |\vec{n}| = \sqrt{(1)^2 + (-4)^2 + (2)^2}$

$\Rightarrow \quad |\vec{n}| = \sqrt{21}$

$\therefore \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|}$

$= \frac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k})$

To calculate directional derivative of $\nabla.\nabla f = \phi$, we shall first calculate $\nabla\phi$.

Now,

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}[12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2] + \hat{j}[24xyz^4 + 48x^3yz^2] + \hat{k}[48xy^2z^3 + 16x^3z^3 + 48x^3y^2z]$$

At (1 – 2, 1)

$$(\nabla g)_{(1,-2,1)} = \hat{i}(48 + 12 + 28.8) + \hat{j}(-48 - 96) + \hat{k}(192 + \quad 92)$$

$$(\nabla g)_{(1,-2,1)} = 48\hat{i} - 144\hat{j} + 400\hat{k}$$

Desired directional derivative = $\nabla\phi.\hat{n}$

$$= (348\hat{i} + 144\hat{j} + 400\hat{k}).\frac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k})$$

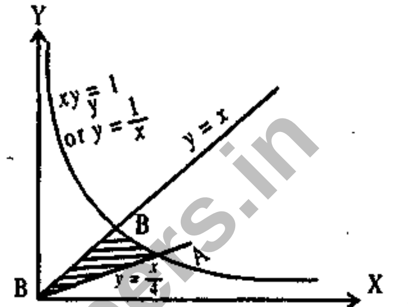
$$= \frac{1}{\sqrt{21}}(348 + 576 + 800) = \frac{1724}{\sqrt{21}}$$

Q. 6. (*b*) Using Green's theorem, find the area of the region in the first quadrant bounded by the curves

$y = x, y = \frac{1}{x}, y = \frac{x}{4}$

Ans. Given curves are $y = x, y = \frac{1}{x}, u = \frac{x}{4}$.

Region bounded by these curves in first quadrant is shown in the following figure.



Required area $= \frac{1}{2}\int_C (xdy - ydx)$, where C represents the boundary of area.

By Green's theorm,

$\Rightarrow$ Required area

$$= \frac{1}{2}\int_{OA}(xdy - ydx) + \frac{1}{2}\int_{AB}(xdy - ydx) + \frac{1}{2}\int_{BO}(xdy - ydx) \quad ...(1)$$

Now,

Along OA, $y = \frac{x}{4}$

$\Rightarrow \quad dy = \frac{dx}{4}$

$\therefore \quad \int_{OA}(xdy - ydx) = \int_{OA}\left(\frac{xdx}{4} - \frac{xdx}{4}\right)$

$\Rightarrow \quad \int_{OA}(xdy - ydx) = 0$

Along AB, $\quad y = \frac{1}{x}$

$$\Rightarrow \quad dy = -\frac{1}{x^2}dx$$

$$\Rightarrow \quad \int_{AB}(xdy - ydx) = \int_{AB} x\left(\frac{-1}{x^2}\right)dx - \frac{1}{x}dx$$

$$= -2\int_{AB}\frac{1}{x}dx$$

To evaluate this integral, we shall first calculate value of x at A and B.

Now,

At A, $\quad y = \frac{x}{4}$ & $y = \frac{1}{x}$

$$\Rightarrow \quad \frac{x}{4} = \frac{1}{x}$$

$$\Rightarrow \quad x^2 = 4$$

$$\Rightarrow \quad x = 2$$

At B,. $\quad y = x$ & $y = \frac{1}{x}$

$$\Rightarrow \quad x = \frac{1}{x}$$

$$\Rightarrow \quad x^2 = 1$$

$$\Rightarrow \quad x = 1$$

$$\Rightarrow \quad \int_{AB}(xdy - ydx) = -2\int_2^1 \frac{1}{x}dx$$

$$= -2[\log x]_2^1$$

$$= 2\log 2 \qquad [\because \log 1 = 0]$$

Also,

Along BO, $\quad y = x$

$$\Rightarrow \quad dy = dx$$

$$\Rightarrow \quad \int_{BO}(xdy - ydx) = \int_{BO} xdx - xdx$$

$$= 0$$

Thus, putting above values in (1),

Required area $= \frac{1}{2}(0) + \frac{1}{2}(2\log 2) + \frac{1}{2}(0)$

$$= \log 2$$

6.(c) Prove that $\left(y^2 - z^2 + 3yz\right)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational.

Ans. Given vector

$$\vec{V} = (y^2 - z^2 + 3yz)\hat{i} + (3xz + 2xy)$$
$$\hat{j}(3xy - 2xz + 2z)\hat{k}$$

Here,

$$V_1 = y^2 - z^2 + 3yz$$
$$V_2 = 3xz + 2xy$$
$$V_3 = 3xy - 2x^2 + 2z$$

For $\vec{V}$ to be irrotational, $\nabla \times \vec{V} = 0$.

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$$

Now,

$\nabla \times \vec{V}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz & 3xz + dxy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \hat{i}[3x - 3x] - \hat{j}[3y - 2z + 2z - 3y]$$
$$+ \hat{k}[3z + 2y - 2y - 3z]$$

$$= 0$$

$\Rightarrow \vec{V}$ is irrotationd.

For $\vec{V}$ to be solenoidal $\nabla.\vec{V} = 0$

Now,

$$\nabla.\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

$$= 0 + 2x - 2x + 2$$
$$= 2$$

$\therefore \nabla.\vec{V} \neq 0$

$\Rightarrow \vec{V}$ is not solenoidal

Hence given question is wrong.

7. (*a*) Changing the order of integration of

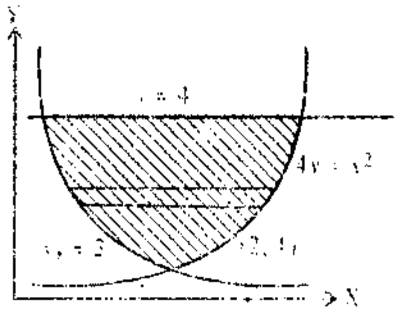
$$\int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dx \, dy$$

Show that $\int_0^\infty \left(\frac{\sin nx}{x}\right) dx = \frac{\pi}{2}$

Ans. See Q.5(*a*) of 2003-04.

7. (*b*) Determine the area bounded by the curves $xy = 2, 4y = x^2$ **and** $y = 4$.

Ans. Region bounded by given curres is shown in followiag figure.



Required area $= \iint_R dx\, dy$

where R is the shaded region.

Here limits of x are $\frac{2}{y}$ to $\sqrt{4y}$ and for y are 2 to 4.

$$\Rightarrow \text{Required area} = \int_{y=1}^{4} \int_{x=\frac{2}{y}}^{2\sqrt{y}} dx\, dy$$

$$= \int_{y=1}^{4} \left(2\sqrt{y} - \frac{2}{y}\right) dy$$

$$= 2\left[\frac{2}{3} y^{3/2} - \log y\right]_{y=1}^{4}$$

$$= 2\left[\frac{2}{3}(4)^{3/2} - \log 4\right]$$

$$-2\left[\frac{2}{3}(4)^{3/2} - \log 2\right]$$

$$= 2\left[\frac{2}{3}(2^2)^{3/2} - \log 2^2\right] - 2\left(\frac{2}{3}\right)$$

$$= 2\left[\frac{2}{3} \cdot 2^3 - 2\log 2\right] - \frac{4}{3}$$

$$= \frac{32}{3} - 2(\log 2^2 - \frac{4}{3}$$

$$= \frac{28}{3} - 4\log 2$$

Q. 7. (*c*) For a β function, show that $\beta(p\, q) = \beta(p+1, q) + \beta(p, q+1)$

Ans. R.H.S $= \beta(p+1, q) + \beta(p, q+1)$

$$= \frac{\overline{|p+1}\,\overline{|q}}{\overline{|p+q+1}} + \frac{\overline{|p}\,\overline{|q+1}}{\overline{|p+q+1}}$$

$$\left[\because \beta(m,n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}\right]$$

$$= \frac{p\overline{|q}\,\overline{|q}}{\overline{|p+q+1}} + \frac{\overline{|p}\, q\overline{|q}}{\overline{|p+q+1}}$$

$$\left[\because \overline{|n+1} = n\overline{|n}\right]$$

$$= (p+q)\frac{\overline{|p}\,\overline{|q}}{\overline{|p+q+1}}$$

$$= \frac{(p+q)\overline{|p}\,\overline{|q}}{(p+q)\overline{|p+q}} \qquad \left[\because \overline{|n+1} = n\overline{|n}\right]$$

$$= \frac{\overline{|p}\,\overline{|q}}{\overline{|p+q}}$$

$$= \beta(p, q) = \text{L.H.S.}$$

Hence proved