B.TECH.

EAS-103

FIRST SEMESTER THEORY 2010-11

MATHEMATICS–I

Time: 3 Hours *Total Marks: 100*

Note: *Q. No. 1 is compulsory and carries 5 marks. Attempt one question from each unit, symbols have their usual meaning.*

SECTION A

Q. 1. All parts of this question are compulsory: **(2 × 10 = 20)**

(a) If $u = f\left(\frac{y}{x}\right)$ **then**

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ...$$

Ans: 0

Justification: u is a homogeneous function of degree 0.

∴ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0u = 0$$

(b) The curve $x^{2/3} + y^{2/3} = a^{2/3}$ **is symmetrical about.......**

Ans: Both x axis and y axis

Justification: Replacing x by $-x$ eqn remains unchanged, therefore curve is symmetrical about y axis. Replacing y by $-y$ eqn remains unchanged, therefore curve is symmetrical about x axis.

Indicate True or False of the following statements:

(c) (i) Two functions u and v are functionally dependent if their Jacobian with respect to x and y is zero. **(True/False)**

(ii) If $f(x,y) = 1 - x^2y^2$, then stationary point is (0, 0). **(True/False)**

Ans: (c) (i) True (ii) True

Justification: $\frac{\partial f}{\partial x} = -2xy^2, \frac{\partial f}{\partial y} = -2x^2y$

$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ gives $(x, y) = (0, 0)$, which is stationary point.

(d) (i) The minimum value of $f(x,y) = x^2 + y^2$ is zero. **(True/False)**

(ii) If u, v are functions of r, s are themselves function of x, y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(x,y)}{\partial(r,s)}$. **(True/False)**

Ans: (d) (i) True (ii) False.

Pick the correct answer of the choices given below:

(e) The eigen values of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **are**

(a) 0, 0, 0 (b) 0, 0, 1 (c) 0, 0, 3 (d) 1, 1, 1

Ans: (e) (c) 0, 0, 3

Justification: For eigen values $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 3-\lambda & 1-\lambda & 1 \\ 3-\lambda & 1 & 1-\lambda \end{vmatrix} = 0$$

Applying $C_1 \Rightarrow C_1 + C_2 + C_3$

$$\Rightarrow (3-\lambda)\begin{vmatrix}1 & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{vmatrix}=0$$

$\Rightarrow \lambda^2 (3-\lambda) = 0$

$\Rightarrow \lambda = 0, 0, 3$

(f) The rank of the matrix $\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 1 & 1\end{bmatrix}$ **is**

(a) 0 (b) 1 (c) 2 (d) 3

Ans: (f) (c) 2

Justification: Given matrix = $\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 1 & 1\end{bmatrix}$

$\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & -1\\ 0 & 0 & 0\end{bmatrix}$ By $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

∴ Rank = No. of non zero rows in echelon form
= 2

(g) $\dfrac{\beta(m+1,n)}{\beta(m,n)}$ **is equal to**

(a) $\dfrac{m}{n}$ **(b)** $\dfrac{m+1}{n}$ **(c)** $\dfrac{m-1}{n}$ **(d)** $\dfrac{m}{m+n}$

Ans: (g) (d) $\dfrac{m}{m+n}$

Justification: $\dfrac{\beta(m+1,n)}{\beta(m,n)} = \dfrac{\dfrac{\overline{|m+1}\,\overline{|n}}{\overline{|m+n+1}}}{\dfrac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}}$

$\because \beta(m,n) = \dfrac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$

$= \dfrac{\overline{|m+1}\,\overline{|n}\,\overline{|m+n}}{\overline{|m}\,\overline{|n}\,\overline{|m+n+1}} = \dfrac{m\overline{|m}\,\overline{|n}\,\overline{|m+n}}{\overline{|m}\,\overline{|n}(m+n)\overline{|m+n}}$

as $\overline{|r+1} = r\overline{|r}$ $= \dfrac{m}{m+n}$

(h) The valoe of the integral $\int_0^\infty e^{-x^2}\,dx$ **is**

(a) $\dfrac{2}{\sqrt{\pi}}$ **(b)** $\dfrac{\sqrt{\pi}}{2}$ **(c)** $\dfrac{\pi}{2}$ **(d)** $\dfrac{2}{\pi}$

Ans. (b) $\dfrac{\sqrt{\pi}}{2}$.

Fill up the blanks with the correct answer:

(i) The Gauss divergence theorem relates certain surface integrals to——
(volume integrals/line integrals)

Ans: Volume integrals

(j) The vector field $\vec{F} = x\hat{i} - y\hat{j}$ is divergence free————
(but not irrotational/and irrotational)

Ans: and irrotational.

Justification: $\nabla \times \vec{F} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x & -y & 0\end{vmatrix} = 0$

$\Rightarrow \vec{F}$ is irrotational.

SECTION B

Q. 2. Attempt any three parts of the following:
(10 × 3 = 30)

(a) If $y = \sin(a\sin^{-1}x)$. Find $(y_n)0$.

Ans. $\because y = \sin(a\sin^{-1}x)$ ...(1)

$\Rightarrow \dfrac{dy}{dx} = \cos(a\sin^{-1}x)\,\dfrac{a}{\sqrt{1-x^2}}$

$\Rightarrow y_1 = \dfrac{a}{\sqrt{1-x^2}}\cos(a\sin^{-1}x)$ ...(2)

$\Rightarrow \sqrt{1-x^2}\;y_1 = a\cos(a\sin^{-1}x)$

$\Rightarrow (1-x^2)y_1^2 = a^2\cos^2(a\sin^{-1}x)$

$= a^2[1-\sin^2(a\sin^{-1}x)]$

$\Rightarrow (1-x^2)y_1^2 = a^2[(1-y^2)]$ $[\because y = \sin(a\sin^{-1}x)]$

Differentiating it again, we get,

$(1-x^2)\,2y_1y_2 - 2xy_1^2 = a^2(-2yy_1)$

$\Rightarrow (1-x^2)y_2 - xy_1 + a^2y = 0$ ...(3)

Differentiating above expression n times by Leibnitz's theorem, we get,

$[(1-x^2)\,y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)\,y_n]$

$-[\,x\,y_{n+1} + n\,C_1\,(1)\,y_n] + a^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} + n(-2x)\,y_{n+1} + \dfrac{n(n-1)}{2}\,(-2)y_n$

$-xy_{n+1} - n\,(1)\,y^n + a^2y_n = 0$

$\left[\because {}^nC_1 = n, {}^nC_2 = \dfrac{n(n-1)}{2}\right]$

$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1}$

$+ (a^2 - n^2 + n - n)\,y_n = 0$

$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1}$

$+ (a^2 - n^2)\,y_n = 0$ ...(4)

Putting $x = 0$ in (1),

$y(0) = \sin(a\sin^{-1}0) = 0$

Putting $x = 0$ in (2),

$y_1(0) = \dfrac{a}{\sqrt{1-0}}\cos(a\sin^{-1}0)$

$\Rightarrow \quad y_1(0) = a\cos 0$

$\Rightarrow \quad y_1(0) = a\ [\therefore \cos 0 = 1]$

Putting $x = 0$ in (3),

$(1-0^2)\,y_2(0) - 0\,y_1(0) + a^2\,y(0) = 0$

$y_2(0) + a^2(0) = 0\ [\therefore y(0) = 0]$

$\Rightarrow \quad y_2(0) = 0$

Thus,

$y_1(0) = a$ and $y_2(0) = 0$

Again, putting $x = 0$ in (4), we get,

$(1-0^2)\,y_{n+2}(0) - (2n+1)(0) + (a^2 - n^2)\,y_n(0) = 0$

$\Rightarrow y_{n+2}(0) = (n^2 - a^2)\,y_n(0)$ ...(5)

Now, two cases arise.

Case I: When n is even

Putting $n = 2$, (5) gives $y_4(0) = (2^2 - a^2)\,y_2(0)$

$= 0\ [\because y_2(0) = 0]$

Putting $n = 4$, (5) gives $y_6(0) = (4^2 - a^2)\,y_4(0)$

$= 0[\therefore y_4(0) = 0]$

Putting $n = 6$, (5) gives, $y_8(0) = (6^2 - a^2)\,y_6(0)$

$= 0\ [\because y_4(0) = 0]$

Thus, $y_n(0) = 0$ if n is even.

Case II. When n is odd

Putting $n = 1$, (5) gives, $y_3(0) = (1^2 - a^2)\,y_1(0)$

$\Rightarrow y_3(0) = (1^2 - a^2)\,a\ [\because y_1(0) = a]$

Putting $n = 3$, (5) gives, $y_5(0) = (3^2 - a^2)\,y_3(0)$

$\Rightarrow y_5(0) = (1^2 - a^2)(3^2 - a^2)\,a$

Putting $n = 5$, (5) gives, $y_7(0) = (5^2 - a^2)\,y_5(0)$

$\Rightarrow y_7(0) = (1^2 - a^2)(3^2 - a^2)(5^2 - a^2)\,a$

$\Rightarrow y_n(0) = (1^2 - a^2)(3^2 - a^2)(5^2 - a^2)$ ...$[(n-2)^2 - a^2]\,a$

if n is odd

$$\text{Thus, } y_n(0) = \begin{cases} 0 \text{ if } n \text{ is even} \\ (1^2 - a^2)(3^2 - a^2)....[(n-2)^2 - a^2]a \\ \text{if } n \text{ is odd and } n \neq 1 \end{cases}$$

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(b) If u, v, w are the roots of the equation. $(x-a)^3\,(x-b)^3 + (x-c)^3 = 0$,

then find $\dfrac{\partial(u,v,w)}{\partial(a,b,c)}$.

Ans: Given equation is

$(x-a)^3 + (x-b)^3 + (x-c)^3 = 0$ ...(1)

Now,

$(x-a)^3 = x^3 - a^3 - 3x^2a + 3xa^2$

$[\because (a-b)^3 = a^2 - b^3 - 3a^2b + 3ab^2]$

$\Rightarrow (x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$

Similarly,

$(x-b)^3 = x^3 - 3bx^2 + 3b^2x - b^3$

$(x-c)^3 = x^3 - 3cx^2 + 3c^2x - c^3$

Putting these values in (1), we get,

$x^3 - 3ax^2 + 3a^2x - a^3 + x^3 - 3bx^2 + 3b^2x - b^3$

$+ x^3 - 3cx^2 + 3c^2x - c^3 = 0$

$\Rightarrow 3x^3 - 3(a+b+c)\,x^2 + 3(a^2 + b^2 + c^2)\,x$

$-(a^2 + b^3 + c^3) = 0$ ...(2)

Now, we know that if α, β and γ are the roots of eqn $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha \beta \gamma = -\frac{d}{a}$$

Here, $\alpha = u$, $\beta = v$, $\gamma = w$

Therefore,

$$u + v + w = -\frac{3(a+b+c)}{-3}$$

$$\Rightarrow u + v + w = a + b + c$$

$$\Rightarrow u + v + w - a - b - c = 0 \qquad ...(3)$$

Also,

$$uv + vw + wu = \frac{3(a^2 + b^2 + c^2)}{3}$$

$$\Rightarrow uv + vw + wu = a^2 + b^2 + c^2$$

$$\Rightarrow uv + vw + wu - a^2 - b^2 - c^2 = 0 \qquad ....(4)$$

Similarly,

$$\Rightarrow uvw = \frac{-(a^3 + b^2 + c^3)}{3}$$

$$\Rightarrow uvw - \frac{(a^3 + b^2 + c^3)}{3} = 0 \qquad ...(5)$$

From (3), (4) and (5)

Let $f_1 = u + v + w - a - b - c$

$f_2 = uv + vw + wu - a^2 - b^2 - C^2$

$f_3 = uvw - \frac{1}{3}(a^3 + b^3 + c^3)$

We know that,

$$\frac{\partial(u,v,w)}{\partial(a,b,c)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(a,b,c)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} \qquad ...(6)$$

Now,

$$\frac{\partial(f_1,f_2,f_3)}{\partial(a,b,c)} = \begin{vmatrix} \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} & \frac{\partial f_1}{\partial c} \\ \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial c} \\ \frac{\partial f_3}{\partial a} & \frac{\partial f_3}{\partial b} & \frac{\partial f_3}{\partial c} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 & -1 \\ -2a & -2b & -2c \\ -a^2 & -b^2 & -c^2 \end{vmatrix}$$

$$= (-1)(-2)(-1)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$\left[\begin{array}{l}\text{Taking } -1 \text{ from } R_1, \\ -R \text{ from } R_2 \text{ and} \\ -\text{from } R_3 \text{ common}\end{array}\right]$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \left[\begin{array}{l}\text{By } C_2 \to C_2 - C_1 \\ \text{and } C_3 \to C_3 - C_1\end{array}\right]$$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$$

$$= -2(b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

Taking $(b - a)$ from C_2 and $(C-a)$ From C_3 common

$$= -2(b-a)(c-a)(c+a-b-a)$$

$$= -2(b-a)(c-a)(c-b)$$

$$= -2(a-b)(b-c)(c-a)$$

$$\Rightarrow \frac{\partial(f_1,f_2,f_3)}{\partial(a,b,c)} = -2(a-b)(b-c)(c-a)$$

Now,

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ v+w & v+w & u+v \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} \text{ by } c_2 \to c_2 - c_1,\; c_3 \to c_3 - c_1$$

$$= (u-v)(u-w)\begin{vmatrix} 1 & 0 & 0 \\ v+w & 1 & 1 \\ vw & w & v \end{vmatrix}$$

Taking $(u-v)$ from c_2 and $(u-w)$ from c_3 common

$= (u-v)(u-w)(v-w)$

$$\Rightarrow \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = -(u-v)(v-w)(w-v)$$

Putting $\frac{\partial(f_1,f_2,f_3)}{\partial(a,b,c)}$ and $\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$ in (6),

we get,

$$\frac{\partial(u,v,w)}{\partial(a,b,c)} = (-1)^3 \frac{2(a-b)(b-c)(c-a)}{-(u-v)(v-w)(w-u)}$$

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(a,b,c)} = -\frac{2(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)} \text{ Ans.}$$

(c) Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Ans. Given matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

characteristic equation of A is

$|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow (8-\lambda)[(7-\lambda)(3-\lambda) - 16] + 6[-6(3-\lambda) + 8]$
$+ 2[24 - 2(7-\lambda)] = 0$

$\Rightarrow (8-\lambda)[21 - 7\lambda - 3\lambda + \lambda^2 - 16] + 6[-18 + 6\lambda + 8]$
$+ 2[24 - 14 + 2\lambda] = 0$

$\Rightarrow (8-\lambda)(\lambda^2 - 10\lambda + 5) + 6(6\lambda - 10) + 2(2\lambda + 10)$
$= 0$

$\Rightarrow 8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda + 36\lambda - 60$
$+ 4\lambda + 20 = 0$

$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\Rightarrow -\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\Rightarrow \lambda(\lambda^2 - 15\lambda - 3\lambda + 45) = 0$

$\Rightarrow \lambda(\lambda - 15) - 3(\lambda - 15)] = 0$

$\Rightarrow \lambda(\lambda - 3)(\lambda - 15) = 0$

$\Rightarrow \lambda = 0, 3, 15$, which are eigen values of A.

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A

corresponding to eigen value λ, then $(A - \lambda I)^x = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(1)$$

For $\lambda = 0$, (1) gives

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 7x_2 + 4x_3 = 0$

$2x_1 - (-4x_2) + 3x_3 = 0$

From first two equations,

$$\frac{x_1}{\begin{vmatrix}-6 & 2\\ 7 & -4\end{vmatrix}} = \frac{x_2}{\begin{vmatrix}8 & 2\\ -6 & -4\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}8 & -6\\ -6 & 7\end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$\Rightarrow$Eigen vector for eigen value 0 is $\begin{bmatrix}1\\2\\2\end{bmatrix}$

Now, for $\lambda = 3$ (1) gives,

$$\begin{bmatrix}5 & -6 & 2\\ -6 & 4 & -4\\ 2 & -4 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow 5x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 4x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 0x_3 = 0$

From first two equations,

$$\frac{x_1}{\begin{vmatrix}-6 & 2\\ 4 & -4\end{vmatrix}} = \frac{x_2}{-\begin{vmatrix}5 & 2\\ -6 & -4\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}5 & -6\\ -6 & 4\end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{+1} = \frac{x_3}{-2}$$

$\Rightarrow$ Eigen vector for eigen value 3 is $\begin{bmatrix}2\\1\\-2\end{bmatrix}$

Again, for $\lambda = 15$ (1) gives,

$$\begin{bmatrix}-7 & -6 & 2\\ -6 & -8 & -4\\ 2 & -4 & -12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow -7x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$

$2x_1 - 4x_2 - 12x_3 = 0$

From first two equations,

$$\frac{x_1}{\begin{vmatrix}-6 & 2\\ -8 & -4\end{vmatrix}} = \frac{x_2}{\begin{vmatrix}-7 & 2\\ -6 & -4\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-7 & -6\\ -6 & -8\end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$\Rightarrow$ Eigen vector for eigen value 15 is $\begin{bmatrix}2\\-2\\1\end{bmatrix}$

Thus, eigen values of A are 0, 3 and 15 corresponding eigen vectors are

$\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\1\\-2\end{bmatrix}$ and $\begin{bmatrix}2\\-2\\1\end{bmatrix}$ respectively

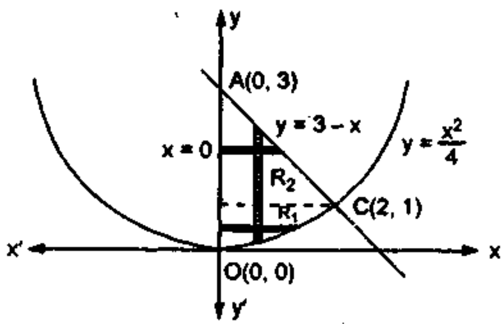
(d) Change the order of integration in

$$I = \int_0^2 \int_{x^2/4}^{3-x} xy\,dy\,dx$$

and hence evaluate it.

Ans: Given integral $I = \int_{x=0}^{2} \int_{y=\frac{x^2}{4}}^{3-x} xy\,dy\,dx$

The region of integration is shown in the figure.



By changing the order of integration,

$$I = \int_{R_1}\int xy\,dx\,dy + \int_{R_2} xy\,dx\,dy$$

For R_1, Limits of x are from 0 to $\sqrt{4y}$

Limits of y are from 0 to 1

For R_2,

Limits of x are from 0 to $3-y$

Limits of y are from 1 to 3.

Thus $I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{4y}} xy\,dx\,dy + \int_{y=1}^{3}\int_{x=0}^{3-y} xy\,dx\,dy$

$$= \int_{y=0}^{1}\left[\frac{x^2y}{2}\right]_{x=0}^{\sqrt{4y}} dy + \int_{y=1}^{3}\left[\frac{x^2y}{2}\right]_{x=0}^{3-y} dy$$

$$= \frac{1}{2}\int_{y=0}^{1} 4y(y)dy + \frac{1}{2}\int_{y=1}^{3}(3-y)^2 y\,dy$$

$$= 2\int_{y=0}^{1} y^2 dy + \frac{1}{2}\int_{y=1}^{3}(9+y^2-6y)y\,dy$$

$$= 2\left[\frac{y^3}{3}\right]_{y=0}^{1} + \frac{1}{2}\int_{y=1}^{3}(y^3-6y^2+9y)dy$$

$$= 2\left(\frac{1}{3}\right) + \frac{1}{2}\left[\frac{y^4}{4} - 6\frac{y^3}{3} + \frac{9y^2}{2}\right]_{y=1}^{3}$$

$$= \frac{2}{3} + \frac{1}{2}\left[\frac{81}{4} - 54 + \frac{81}{2} - \frac{1}{4} + \frac{6}{3} - \frac{9}{2}\right]$$

$$= \frac{2}{3} + \frac{1}{2}\left[\left(\frac{81}{4} - \frac{1}{4}\right) + \left(\frac{81}{2} - \frac{9}{2}\right) - 54 + 2\right]$$

$$= \frac{2}{3} + \frac{1}{2}[20+36-52] = \frac{2}{3} + \frac{1}{2}(4)$$

$$= \frac{8}{3} \text{ Ans.}$$

(e) Find the volume enclosed between the two surfaces $Z = 8 - x^2 - y^2$ and $Z = x^2 + 3y^2$.

Ans : Required volume $= \iiint_v dx\,dy\,dz$, where v is the given region.

For v,

Limits of z are from $x^2 + 3y^2$ to $8 - x^2 - y^2$. For limits of y, we equate z from given surfaces.

$\Rightarrow 8 - x^2 - y^2 = x^2 + 3y^2$

$\Rightarrow 2x^2 + 4y^2 = 8$

$\Rightarrow x^2 + 2y^2 = 4$

Thus, limits of y are from $-\sqrt{\frac{4-x^2}{2}}$ to $+\sqrt{\frac{4-x^2}{2}}$.

Limits of x are from -2 to $+2$.

$\therefore$ Required volume

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{z=x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}(8-x^2-y^2-x^2-3y^2)dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}(8-2x^2-4y^2)dy\,dx$$

Putting $x = 2r\cos\theta$, $y = \sqrt{2}\,r\sin\theta$

$dx\,dy = 2\sqrt{2}\,r\,dr\,d\theta$

r varies from 0 to 1

θ varies from 0 to 2π.

Required volume

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}[8-2(4r^2)]2\sqrt{2}\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}8(1-r^2)2\sqrt{2}\,r\,dr\,d\theta$$

$$= 16\sqrt{2}\int_{0}^{2\pi}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{1} d\theta$$

$$= 16\sqrt{2}\int_{\theta=0}^{2\pi}\frac{1}{4}d\theta$$

$$= 4\sqrt{2}(2\pi)$$

$$= 8\sqrt{2}\pi \text{ Ans.}$$

SECTION C

Attempt any two parts from each equation. All questions are compulsory. (5 × 2 × 5 = 50)

Q. 3. (a) Trace the curve $y^2(2a-x)=x^3$.

Ans: The euqation of the curve is

$$y^2(2a-x)=x^3 \quad ...(1)$$

1. Symmetry: Since (1) contains only even powers of y, the curve is symmetrical about x axis.

2. Origin: The tangents at the origin are given by $y^2=0$, i.e., $y=0$, $y=0$. Since the two tangents are real and coincident, origin is a cusp.

3. Asymptotes: Equating to zero, the coefficient of y^2, the highest degree term in y, the asymptote parallel to y axis is $x-2a=0$ i.e., $x=2a$. There is no other asymptote of the curve.

4. Points of intersection: The curve meets x axis and y axis at the origin only.

5. Region: From (1), $y = x\sqrt{\dfrac{x}{2a-x}}$

When $x<0$, y is imaginary

$\Rightarrow$ No portion of the curve lies to the left of the line $x=0$ i.e. y axis.

When $0<0<2a$, y is real.

When $x>2a$, y is imaginary

$\Rightarrow$ No portion of curve lies to the right of the line $x=2a$.

6. Special points: From (1),

$$y=\frac{y^{3/2}}{\sqrt{2a-x}} \quad ...(2)$$

$$\therefore \quad \frac{dy}{dx}=\frac{\sqrt{x}(3a-x)}{(2a-x)^{3/2}}$$

$$\therefore \quad \frac{dy}{dx}=0$$

When $\sqrt{x}(3a-x)=0$

$\Rightarrow \quad x=0, x=3a$

Here $x=3a$ is not possible, because when $x=3a$, from (2), y is imaginary.

When $x=0$, $y=0$.

$\therefore$ Tangent at (0, 0) i.e. at origin is parallel to x axis i.e. the tangent at origin is x axis.

Again $\dfrac{dy}{dx}\to\infty$ when $x\to 2a$. From (2), $x\to 2a$, $y\to\infty$. Thus, $x=2a$ is an asymptote.

When $0<x<2a$, $\dfrac{dy}{dx}$ is positive.

$\therefore$ For positive values of y, y is an increasing function of x, i.e. the curve rises for values of x between 0 and $2a$.

(b) If $Z=f(x+ct)+\varphi(x-ct)$ show that $\dfrac{\partial^2 z}{\partial t^2}=c^2\dfrac{\partial^2 z}{\partial x^2}$.

Ans. $z=f(x+ct)+\phi(x-ct)$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{\partial}{\partial x}[f(x+ct)]+\frac{\partial}{\partial x}[\phi(x-ct)]$$

$$\Rightarrow \frac{\partial z}{\partial x}=f'(x+ct)+\phi'(x-ct)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}[f'(x+ct)]+\frac{\partial}{\partial x}[\phi'(x-ct)]$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}=f''(x+ct)+\phi''(x-ct)$$

$$\Rightarrow c^2\frac{\partial^2 z}{\partial x^2}=c^2 f''(x+ct)+c^2\phi''(x-ct) \quad ...(1)$$

$$\text{Also,}\ \frac{\partial z}{\partial t}=\frac{\partial}{\partial t}[f(x+ct)]+\frac{\partial}{\partial t}[\phi(x-ct)]$$

$$\Rightarrow \frac{\partial z}{\partial t}=f'(x+ct)+\frac{\partial}{\partial t}(ct)+\phi'(x-ct).\frac{\partial}{\partial t}(-ct)$$

$$\Rightarrow \frac{\partial z}{\partial t}=cf'(x+ct)-c\phi'(x-ct)$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2}=\frac{\partial}{\partial t}[cf'(x+ct)]-\frac{\partial}{\partial t}[c\phi'(x-ct)]$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2}=cf''(x+ct)\frac{\partial}{\partial t}(ct)-c\phi''(x-ct)\frac{\partial}{\partial t}(-ct)$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2}=c^2f''(x+ct)+c^2\phi''(x-ct) \quad ...(2)$$

From (1) and (2),

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Hence proved.

(c) Expand e^{ax} sin by in the powers of x and y as far as terms of third degree.

Ans. Given function $f = e^{ax}$ sin by

$$\frac{\partial f}{\partial x} = ae^{ax} \sin by$$

$$\frac{\partial f}{\partial y} = be^{ax} \cos by$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(ae^{ax} \sin by)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = a^2 e^{ax} \sin by$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(be^{ax} \cos by)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x \partial y} = abe^{ax} \cos by \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(be^{ax} \cos by)$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial y^2} = -b^2 e^{ax} \sin by$$

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right)$$

$$\Rightarrow \quad \frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}(a^2 e^{ax} \sin by)$$

$$\Rightarrow \frac{\partial^3 f}{\partial x^3} = a^3 e^{ax} \sin by \quad \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right)$$

$$\Rightarrow \quad \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}(abe^{ax} \cos by)$$

$$\Rightarrow \quad \frac{\partial^3 f}{\partial x^2 \partial y} = a^2 be^{ax} \cos by$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)$$

$$\Rightarrow \quad \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}(-b^2 e^{ax} \sin by)$$

$$\Rightarrow \frac{\partial^3 f}{\partial x \partial y^2} = -ab^2 e^{ax} \sin by \quad \frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right)$$

$$\Rightarrow \quad \frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y}(-b^2 e^{ax} \sin by)$$

$$\Rightarrow \frac{\partial^3 f}{\partial y^3} = -b^3 e^{ax} \cos by$$

Now, we make the following table:

Function	Expression	Value at (0, 0)
f	e^{ax} sin by	0
$\frac{\partial f}{\partial x}$	ae^{ax} sin by	0
$\frac{\partial f}{\partial y}$	be^{ax} cos by	b
$\frac{\partial^2 f}{\partial x^2}$	$a^2 e^{ax}$ sin by	0
$\frac{\partial^2 f}{\partial x \partial y}$	abe^{ax} cos by	ab
$\frac{\partial^2 f}{\partial y^2}$	$-b^2 e^{ax}$ sin by	0
$\frac{\partial^3 f}{\partial x^3}$	$a^3 e^{ax}$ sin by	0
$\frac{\partial^3 f}{\partial x^2 \partial y}$	$a^2 be^{ax}$ cos by	$a^2 b$
$\frac{\partial^3 f}{\partial x \partial y^2}$	$-ab^2 e^{ax}$ sin by	0
$\frac{\partial^3 f}{\partial y^3}$	$-b^3 e^{ax}$ cos by	$-b^3$

By Taylor's theorem, expansion of $f(x, y)$ about the point (h, k)

$$f(x, y) = f(h, k) + \frac{1}{1!}\left[(x-h)\frac{\partial f}{\partial x} + (y-k)\frac{\partial f}{\partial y}\right]$$

$$+ \frac{1}{2!}\left[(x-h)^2\frac{\partial^2 f}{\partial x^2} + 2(x-h)(y-k)\frac{\partial^2 f}{\partial x \partial y} + (y-k)^2\frac{\partial^2 f}{\partial y^2}\right]$$

$$+ \frac{1}{3!}[(x-h)^3\frac{\partial^3 f}{\partial x^3} + 3(x-h)^2(y-k)\frac{\partial^3 f}{\partial x^2 \partial y}$$

$$+ 3(x-h)(y-k)^2\frac{\partial^3 f}{\partial x \partial y^2} + (y-k)^3\frac{\partial^3 f}{\partial y^3}] + \ldots.$$

Here, $h = 0$, $k = 0$

$$\Rightarrow f(x, y) = f(0, 0) + \frac{1}{1!}\left[x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right] + \frac{1}{2!}\left[x^2\frac{\partial^2 f}{\partial x^2}\right.$$

$$\left. + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right] + \frac{1}{3!}\left[x^3\frac{\partial^3 f}{\partial x^3} + 3x^2y\frac{\partial^3 f}{\partial x^2 \partial y}\right.$$

$$\left. + 3xy^2\frac{\partial^3 f}{\partial x \partial y^2} + y^3\frac{\partial^3 f}{\partial y^3}\right] + \ldots.$$

$$\Rightarrow f(x, y) = \frac{1}{1!}[by] + \frac{1}{2!}[ab(2xy)] + \frac{1}{3!}\left[3x^2y(a^2b)\right.$$

$$- b_3y_3] + \ldots..$$

$$\Rightarrow f(x_1 y) = \text{by} + \text{abxy} + \frac{1}{2}a^2bx^2y - \frac{b^3}{6}y^3 + \ldots..$$

Q. 4. (a) A rectangular box, open at the top, is to have a volume of 32 cubic feet. Determine the dimensions of the box requiring least material for its construction.

Ans: Let x, y and z be the length, breadth and height of the box respectively.

$\because$ volume of the box is 32 cubic feet

$\Rightarrow xyz = 32$ – (1)

[$\therefore$ volume = length × breadth × height]

Let s be the surface area of the box.

$\Rightarrow$ s = Total surface area of closed box – area of the top.

$= 2(xy + yz + zx) - xy$

[$\therefore$ surface area = $2(lb + bh + hl)$]

$= xy + 2yz + 2zx$

Here s is to be minimized

$\therefore$ Let $f = xy + 2yz + 2zx$

and $\phi = xyz - 32$ [From (1)]

Lagrange's equations are,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$

$\Rightarrow \quad y + 2z + \lambda yz = 0$ ...(2)

$x + 2z + \lambda xz = 0$ ...(3)

$2y + 2z + \lambda xy = 0$ ...(4)

From (2), $\lambda = -\frac{y + 2z}{yz}$ ...(5)

From (3), $\lambda = -\frac{x + 2z}{xz}$ ...(6)

From (4), $\lambda = -\frac{2x + 2y}{xy}$ ...(7)

Equating λ from (5) and (6), we get,

$$-\frac{y + 2z}{yz} = -\frac{x + 2z}{xz}$$

$\Rightarrow x(y + 2z) = y(x + 2z)$

$\Rightarrow xy + 2xz = xy + 2yz$

$\Rightarrow \quad 2xz = 2yz$

$\Rightarrow \quad x = y$

Again, equating λ from (5) and (7), we get,

$$-\frac{y + 2z}{yz} = \frac{2x + 2y}{xy}$$

$\Rightarrow x(y + 2z) = z(2z + 2y)$

$\Rightarrow xy + 2xz = 2xz + 2yz$

$\Rightarrow \quad xy = 2yz$

$\Rightarrow \quad z = \frac{x}{2}$

Putting $y = x$ and $z = \frac{x}{2}$ in (1), we get,

$$(x)(x)\left(\frac{x}{2}\right) = 32$$

$\Rightarrow \quad x^3 = 64$

$\Rightarrow \quad x = 4$

$\Rightarrow \qquad y = 4 \qquad [\because y = x]$

$\Rightarrow \qquad z = 2 \qquad [\because z = \frac{x}{2}]$

Thus, length, breadth and height of the box are 4 feet, 4 feet and 2 feet respectively, requiring least material for the construction of the box.

(b) If $u_1 = \frac{x_2x_3}{x_1}$, $u_2 = \frac{x_3x_1}{x_2}$ and $u_3 = \frac{x_1x_2}{x_3}$ find the value of $\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}$.

Ans: We have

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_1x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2}\cdot\frac{1}{x_2^2}\cdot\frac{1}{x_3^2}\begin{vmatrix} -x_2x_3 & x_1x_3 & x_1x_2 \\ x_2x_3 & -x_1x_3 & x_1x_2 \\ x_2x_3 & x_1x_3 & -x_1x_2 \end{vmatrix}$$

[Taking $\frac{1}{x_1^2}$ from R_1, $\frac{1}{x_2^2}$ from R_2 and $\frac{1}{x_3^2}$ from R_3 common]

$$= \frac{(x_2x_3)(x_1x_3)(x_1x_2)}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

[Taking x_2x_3 from C_1, x_1x_3 from C_2 and x_1x_2 from C_3 common]

$$= \frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= -1[1-1] - 1[-1-1] + 1[1+1]$

$= 4$

(c) Find the percentage of error in calculating the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when error of + 1% is made in measuring the major and minor axes.

Ans: Eq^n of given ellipse is

$$\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$$

$\therefore$ error in major axis and minor axis are 1%.

$$\Rightarrow \qquad \frac{\delta a}{a} = \frac{1}{100} \text{ and } \frac{\delta b}{b} = \frac{1}{100}$$

Let A respresents the area of ellipse. To find error in A, we shall express A in terms of a and b, because errors in a and b are known to us. We know that

$A = \pi ab$ [$\therefore$ area of ellipse = πxy where x and y are semimajor and semiminor axis]

Taking log of both sides, we get,

$$\log A = \log(\pi ab)$$

$$\Rightarrow \log A = \log \pi + \log a + \log b$$

$$[\because \log(mn) = \log m + \log n]$$

By differentials, we get,

$$\frac{\delta A}{A} = 0 + \frac{\delta a}{a} + \frac{\delta b}{b}\left[\begin{array}{l}\therefore \pi \text{ is constant} \\ \therefore \text{its differential is zero}\end{array}\right]$$

$$\Rightarrow \qquad \frac{\delta A}{A} = \frac{\delta a}{a} + \frac{\delta b}{b}$$

$$\Rightarrow \frac{\delta A}{A} = \frac{1}{100} + \frac{1}{100}\left[\because \frac{\delta a}{a} = \frac{1}{100}, \frac{\delta b}{b} = \frac{1}{100}\right]$$

$$\Rightarrow \qquad \frac{\delta A}{A} = \frac{2}{100}$$

Thus, error in area of ellipse is 2%.

Q. 5. (a) Test for consistency and solve the following system of equations

$2x - y + 3z = 8$

$-x + 2y + z = 4$

$3x + y - 4z = 0$

Ans: Given system of equations is

$2x - y + 3z = 8$

$-x + 2y + z = 4$

$3x + y - 4z = 0$

$$\text{Augmented matrix } [A:B] = \begin{bmatrix} 2 & -1 & 3 & \vdots & 8 \\ -1 & 2 & 1 & \vdots & 4 \\ 3 & 1 & -4 & \vdots & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \vdots & 4 \\ 2 & -1 & 3 & \vdots & 8 \\ 3 & 1 & -4 & \vdots & 0 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \vdots & 4 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 7 & -1 & \vdots & 12 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \vdots & 4 \\ 0 & -1 & 11 & \vdots & 20 \\ 0 & 7 & -1 & \vdots & 12 \end{bmatrix} R_2 \to 2R_2 - R_3$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \vdots & 4 \\ 0 & -1 & 11 & \vdots & 20 \\ 0 & 0 & 76 & \vdots & 152 \end{bmatrix} R_3 \to R_3 + 7R_2$$

which is in echelon form.

Here, rank of $A = 3$

Rank of $[A: B] = 3$

No of unknowns = 3

$\Rightarrow$ Rank of A = Rank of $[A : B]$ = No. of unknowns

$\Rightarrow$ System is consistent with unique solution. Rewriting the system,

$-x + 2y + z = 4$ ...(1)

$-y + 11z = 20$ ...(2)

$76z = 152$ ...(3)

From (3) $z = \dfrac{152}{76} = 2$

Put $z = 2$ in (2), we get

$-y + 11(2) = 20$

$\Rightarrow \quad -y = -2 \Rightarrow y = 2$

Putting $y = 2$, $z = 2$ in (1), we get,

$-x + 2(2) + 2 = 4$

$\Rightarrow \quad -x + 6 = 4 \Rightarrow x = 2$

Thus, $x = 2, y = 2, z = 2$ **Ans.**

(b) Reduce the following matrix to normal form and hence find its rank:

$$\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

Ans: Given matrix $= \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix} R_3 \to R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix} \begin{matrix} c_2 \to c_2 + c_1 \\ c_3 \to c_3 - 2c_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix} R_3 \to R_3 - 8R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -12 & -4 \end{bmatrix} \begin{matrix} c_3 \to c_3 - 2c_2 \\ c_4 \to c_4 - c_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix} c_3 \to \frac{c_3}{-12}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} c_4 \to c_4 + 4c_3$$

$\sim [I_3 \, O]$

which is in normal form.

$\therefore$ Rank = order of identity matrix in normal form $= 3$.

(c) Show that the matrix

$$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$$

is unitary if and only if $a_2 + b_2 + c_2 + d_2 = 1$.

Ans: Given matrix $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$. Now $\bar{A} = \begin{bmatrix} a-ic & -b-id \\ b-id & a+ic \end{bmatrix}$ [Replacing i by $-i$]

$$\Rightarrow \qquad A^{\theta} = (\bar{A})^T \Rightarrow A^{\theta} = \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$$

Now $$AA^{\theta} = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}\begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$$

$$\Rightarrow AA^{\theta} = \begin{bmatrix} (a+ic)(a-ic)+(-b+id)(-b-id) & (a+ic)(b-id)+(-b+id)(a+ic) \\ (b+id)(a-ic)+(a-ic)(-b-id) & (b+id)(b-id)(a-ic)(a+ic) \end{bmatrix}$$

$$= \begin{bmatrix} a^2-i^2c^2+b^2-i^2d^2 & (a+ic)(b-id)-(b-id)(a+ic) \\ (b+id)(a-ic)-(a-ic)(b+id) & b^2-i^2d^2+d^2-i^2c^2 \end{bmatrix} = \begin{bmatrix} a^2+c^2+b^2+d^2 & 0 \\ 0 & b^2+d^2+a^2+c^2 \end{bmatrix}$$

Now above matrix is identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if and only if $a^2 + b^2 + c^2 + d^2 = 1$

i.e. $AA^{\theta} = I$ if and only if $a^2 + b^2 + c^2 + d^2 = 1$ i.e. A is unitary if and only if $a^2 + b^2 + c^2 + d^2 = 1$

Q. 6. (a) Prove that $\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$.

Ans: Given integral $I = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Put $x^4 = \tan^2\theta$

$$\Rightarrow \qquad x = \sqrt{\tan\theta} \Rightarrow dx = \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\, d\theta$$

at $x = 0$, $\tan\theta = 0 \Rightarrow \theta = 0$, at $x = 1$, $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

Thus, $I = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{\sec\theta} \cdot \frac{1}{\sqrt{\tan\theta}}\sec^2\theta\, d\theta$ $\quad [\because 1 + \tan^2\theta = \sec^2\theta]$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sqrt{\cot\theta}\sec\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}} \frac{1}{\cos\theta} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}} = \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{2\sin\theta\cos\theta}}$$

$$= \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\sin 2\theta}} = \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{4}} \frac{1}{2}\frac{d\theta}{\sqrt{\sin\phi}}$$

[Put $2\theta = \phi$
$\Rightarrow d\theta = \frac{d\phi}{2}$
at $\theta = 0, \phi = 0$
at $\theta = \frac{\pi}{4}, \phi = \frac{\pi}{2}$]

$$= \frac{\sqrt{2}}{4}\int_0^{\pi/2} \sin^{-1/2}\phi\, d\phi$$

$$= \frac{\sqrt{2}}{4}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\phi \cos^{0}\phi\, d\phi = \frac{\sqrt{2}}{4}\,\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)}$$

$$\left[\because \int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}\right]$$

$$= \frac{\sqrt{2}}{8}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{2}}{8}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$= \frac{\sqrt{2}}{8}\beta\left(\frac{1}{4},\frac{1}{2}\right)\left[\because \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\right]$$

$$= \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

Hence proved.

(b) Evaluate $\iiint x^{l-1}y^{m-1}z^{n-1}\,dxdydz$, where $x>0, y>0$ under the condition

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \le 1.$$

Ans: Given integral $I = \iiint_v x^{l-1}y^{m-1}z^{n-1}\,dxdydz$

where v is the region $x>0, y>0, z>0$

and $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \le 1$

Put $\left(\frac{x}{a}\right)^p = X, \left(\frac{y}{b}\right)^q = Y, \left(\frac{z}{c}\right)^r = Z$

$$\Rightarrow \frac{x}{a} = X^{\frac{1}{p}},\ \frac{y}{b} = Y^{\frac{1}{q}},\ \frac{z}{c} = Z^{\frac{1}{r}}$$

$$\Rightarrow x = a\,X^{\frac{1}{p}},\ y = b\,Y^{\frac{1}{q}},\ z = c\,Z^{\frac{1}{r}}$$

$$\Rightarrow dx = \frac{a}{p}X^{\frac{1}{p}-1}dX\ dy = \frac{b}{q}Y^{\frac{1}{q}-1}dY\ dz = \frac{c}{r}Z^{\frac{1}{r}-1}dZ$$

Putting above values,

$$I = \iiint_{v'}\left(aX^{\frac{1}{p}}\right)^{l-1}\left(bY^{\frac{1}{q}}\right)^{m-1}\left(cZ^{\frac{1}{r}}\right)^{n-1}\frac{a}{b}\cdot\frac{b}{q}\cdot\frac{e}{r}\cdot$$

$$X^{\frac{1}{p}-1}\,Y^{\frac{1}{q}-1}\,Z^{\frac{1}{r}-1}\,dZdYdZ$$

where v is the region $Z>0, Y>0, Z>0$ and $Z+Y+Z \le 1$.

$$\Rightarrow I = \frac{1}{pqr}\iiint_{v'} a^{l-1}X^{\frac{l-1}{p}}\,b^{m-1}Y^{\frac{m-1}{q}}\,c^{n-1}Z^{\frac{n-1}{r}}\,abcX^{\frac{1}{p}-1}$$

$$Y^{\frac{1}{q}-1}\,Z^{\frac{1}{r}-1}\,dXdYdZ$$

$$= \frac{a^l b^m c^n}{pqr}\iiint_{v'} X^{\frac{l-1}{p}+\frac{1}{p}-1}\,Y^{\frac{m-1}{q}+\frac{1}{q}-1}\,Z^{\frac{n-1}{r}+\frac{1}{r}-1}\,dXdYdZ$$

$$= \frac{a^l b^m c^n}{pqr}\iiint_{v'} X^{\frac{l}{p}-1}\,Y^{\frac{m}{q}-1}\,Z^{\frac{n}{r}-1}\,dXdYdZ$$

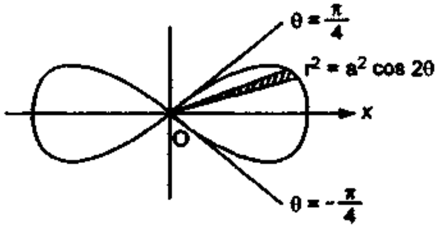
$$= \frac{a^l b^m c^n}{pqr}\,\frac{\Gamma\frac{l}{p}\,\Gamma\frac{m}{q}\,\Gamma\frac{n}{r}}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{b}{r}+1\right)}$$ Ans.

[∴ By Dirichlet's theorem,

$$\iiint_v x^{l-1}y^{m-1}z^{n-1}\,dxdydz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}\Bigg]$$

(c) Find the area of one loop of the lemniscates $r^2 = a^2\cos 2\theta$.

Ans: The given lemniscate is shown in the following figure.



Required area = $\iint\limits_R r\,dr\,d\theta$ where R is the region bounded by one loop of lemniscate $r^2 = a^2 \cos 2\theta$.

Limits of θ are 0 to $a\sqrt{\cos 2\theta}$

Limits of r are $-\dfrac{\pi}{4}$ to $\dfrac{\pi}{4}$.

Thus, required area = $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$

$$= \int_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_{r=0}^{a\sqrt{\cos 2\theta}} d\theta \quad = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2\cos^2\theta\,d\theta$$

$$= \frac{a^2}{2}\left[\frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\because \int \cos ax\,dx = \frac{\sin ax}{a}\right]$$

$$= \frac{a^2}{4}\left[\sin\frac{\pi}{2} - \sin(-\frac{\pi}{2})\right]$$

$$= \frac{a^2}{4}[1+1] = \frac{a^2}{2} \text{ Ans.}$$

Q. 7. (a) Find the directional derivative of $\varphi(x, y, z) = xy^2 + yz^3$ at the point (2, −1, 1) in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at (2, −1, 1)

Ans: Given surface $\phi = xy^2 + yz^3$

Now $\nabla\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

$\Rightarrow \nabla\phi = \hat{i}(y^2) + \hat{j}(2xy + z^3) + \hat{k}(3yz^2)$

$\Rightarrow (\nabla\phi)_{(2,-1,+1)} = \hat{i} + \hat{j}(-4+1) + \hat{k}(-3)$

$\Rightarrow (\nabla\phi)_{(2,-1,-1)} = \hat{i} - 3\hat{j} - 3\hat{k}$

∴ We want to find normal to the surface

$x \log z - y^2 + 4 = 0$

∴ Let $g = x \log z - y^2 + 4$

$\Rightarrow \nabla g = \dfrac{\partial g}{\partial x}\hat{i} + \dfrac{\partial g}{\partial y}\hat{j} + \dfrac{\partial g}{\partial z}\hat{k}$

$\Rightarrow \nabla g = \log z\ \hat{i} - 2y\hat{j} + \dfrac{x}{z}\hat{k}$

$\Rightarrow (\nabla g)_{(2,-1,1)} = \log 1\ \hat{i} + 2\hat{j} + \dfrac{2}{1}\hat{k}$

$\Rightarrow \quad \vec{n} = 2\hat{j} + 2\hat{k}$

[∵ Log 1 = 0 and $\vec{n} = \nabla g$ represent normal to surface]

$\Rightarrow \quad \hat{n} = \dfrac{\vec{n}}{|\vec{n}|}$

$\Rightarrow \quad \hat{n} = \dfrac{1}{\sqrt{2^2 + 2^2}}(2\hat{j} + 2\hat{k})$

$$\left[\begin{array}{l}\because \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \\ \Rightarrow |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}\end{array}\right] = \frac{1}{2\sqrt{2}}(2\hat{j} + 2\hat{k})$$

$\Rightarrow \quad \hat{n} = \dfrac{1}{\sqrt{2}}(\hat{j} + \hat{k})$

Now, Desired directional derivative = $\nabla\phi.\hat{n}$

$$= (1 - 3\hat{j} - 3\hat{k}) \cdot \frac{1}{2}(\hat{j} + \hat{k}) = \frac{1}{\sqrt{2}}(-3-3)$$

$$\left[\begin{array}{l}\because \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \\ \Rightarrow \vec{a} - \vec{b} = a_1b_1 + a_2b_2 + q_3b_3\end{array}\right]$$

$$= \frac{-6}{\sqrt{2}} = \frac{-6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = -3\sqrt{2} \text{ Ans.}$$

(b) If all second order derivatives of φ and $\vec{v}$ are continuous, then show that

(i) curl (grad φ) = $\vec{v}$

(ii) div (curl $\vec{v}$) = 0

Ans: (i) To show that curl (grad φ) = 0, we shall first find grad φ.

Now, grad φ = ∇φ

$\Rightarrow$ grad $\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

Now curl $\vec{v} = \nabla \times \vec{v}$

$$\Rightarrow \quad = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$\Rightarrow$ curl (grad ϕ) = $\nabla \times (\nabla\phi)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right] + \text{two similar terms}$$

$$= \hat{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right] + \text{two similar terms} = 0$$

$\Rightarrow$ curl (grad ϕ) = 0

Hence proved.

(ii) To show that div (curl $\vec{v}$) = 0, we shall first find curl $\vec{v}$.

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\text{curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$\Rightarrow \text{curl } \vec{v} = \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \hat{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

Now, div $\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$\Rightarrow \text{div (curl } \vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 V_3}{\partial x\partial y} - \frac{\partial^2 V_2}{\partial x\partial z} + \frac{\partial^2 V_1}{\partial y\partial z} - \frac{\partial^2 V_3}{\partial x\partial y}$$

$$= \frac{\partial^2 V_2}{\partial x\partial z} - \frac{\partial^2 V_1}{\partial y\partial z} = 0$$

$\Rightarrow$ div (curl $\vec{v}$) = 0 **Hence Proved.**

(c) Find the work done by the force

$$\vec{f} = (2y + 3)\hat{i} + xz\hat{j} + (yz - x)\hat{k}$$

when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2$, $y = t$ and $z = t^3$,

Ans. Required work done = $\int_c \vec{f}.d\vec{r}$

$$= \int_c [(dy + 3)dx + xzdy + (yz - x)dz] \quad \ldots(1)$$

$$\left[\because \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}, \vec{dr} = dx\hat{i} + dy\hat{j}\ dz\hat{k} \Rightarrow \vec{F}\cdot\vec{dr} = F_1dx + F_2dy + F_3dz\right]$$

Along c,

$x = 2t^2 \cdot y = t \cdot z = t^3$

$\Rightarrow dx = 4tdt,\ dy = dt \quad dz = 3t^2dt$

t varies from 0 to 1.

Putting these values in (1), we get,

Required work done

$$= \int_{t=0}^{1}(2t + 3)4tdt + \int_{t=0}^{1}(2t^2)(t^3)dt + \int_{t=0}^{1}(t.t^3 - 2t^2)3t^2dt$$

$$= \int_{t=0}^{1}(8t^2 + 12t + 2t^5 + 3t^6 - 6t^4)dt$$

$$= \left[\frac{8t^3}{3} + 12\frac{t^2}{2} + 2\frac{t^6}{6} + \frac{3t^7}{7} - \frac{6t^5}{5}\right]_{t=0}^{1}$$

$$= \frac{8}{3} + \frac{12}{2} + \frac{2}{6} + \frac{3}{7} - \frac{6}{5}$$

$$= \frac{560 + 1260 + 70 + 90 - 252}{210}$$

$$= \frac{1728}{210} = \frac{288}{35} \textbf{ Ans.}$$