B.Tech.

SECOND SEMESTER EXAMINATION, 2008-09

MATHEMATICS-II

(EAS-203)

Time : 3 Hours] *[Total Marks : 100*

SECTION-A

Note : **(1) Attemp all qnestions. All questions carry equal marks.**

Fill up the appropriate answers in the space provided :

1. (a) The differential equation

$\left(\frac{d^3y}{dx^3}\right)^4 - 6x^2\left(\frac{dy}{dx}\right)^2 + e^x = \sin xy$ is of .... order and .... degree.

Ans. order = 3 , degree = 4

(b) A particular solution of the differential equation $\frac{dy}{dt} = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$ is ....

Ans. $3(y-1) = e^t(y+1)$

(c) The general solution of the differential equation $\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0$ is given by ....

Ans. $y = (c_1 + c_2x + c_3x^2) + c_4e^x + c_5e^{-x}$

Pick up the Correct Answer from the following :

(d) The Rodrigues formula for Legendre Polynomial $P_n(x)$ is given by

(i) $P_n(x) \frac{1}{\lfloor n \, 2^n} \frac{d^n}{dx^n}(x^2-1)^n$ (ii) $P_n(x) = \frac{\lfloor n}{2^n} \frac{dn}{dx^n}(x^2-1)^{n-1}$

(iii) $P_n(x) = \frac{\lfloor n}{2^{n-1}} \frac{d^n}{dx^n}(x^2-1)^{n-1}$ (iv) $P_n(x) = \frac{1}{\lfloor n \, 2^n}(x^2-1)^n$

Ans. (i) $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

(e) The Laplace transform of the function

$f(t) \begin{cases} 1, & 0 \le t < 2 \\ -1, & 2 \le t < 4, \end{cases}$ $f(t+4) = f(t)$ is given as

(i) $\frac{1 - e^{-2s}}{s(1 + e^{-2s})}$

(ii) $\frac{1+e^{-2s}}{s(1+e^{-2s})}$ (iii) 0 (iv) $\frac{s+1}{s-1}$

Ans. (i) $\frac{1-e^{-2s}}{s(1+e^{-2s})}$

(f) The inverse Laplace transform of $\log\left(\frac{s+1}{s-1}\right)$ is given by

(i) $\frac{2}{t}\cosh t$ (ii) $\frac{2}{t}\sin ht$ (iii) $2t\cos t$ (iv) $2t\sin t$

Ans. (ii) $\frac{2}{t}\sin ht$

Indicate True/False for tbe statements made therein -

(g) (i) A function $f(x)$ is even if $f(-x) = -f(x)$. (True/False)

(ii) A function $f(x)$ is odd if $f(-x) = f(x)$. (True/False)

(iii) Most functions are neither even nor odd.

(True / False)

(iv) A function $f(x)$ can always he expressed as an arithmetic mean of an even und odd function as

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}f(x) - f(-x)]$$

(True/False)

Ans. (i) False (ii) False (iii) True (iv) True.

(h) (i) Witb usual symbols, the PDE

$u_{xx} + u^2 u_{yy} = f(xy)$ is non-linear in 'u' and is of second order. (True/False)

(ii) The small transverse vihrations of a string are governed by one dimensional .eat equation $y_t = a^2 y_{xx}$. (True/False)

(iii) Two dimensional steady state heat flow is given by Laplace's equation $_t = a^2(u_{xx} + u_{tt})$. (True/False)

(iv) The PDE of all sphere whose centre lie on z-axis and given hy equations $^2 + y^2 + (z-a)^2 = b^2$, a and b being constants, are governed hy $x z_y - y z_x = 0$. [rue/False)

Ans. (i) True (ii) False (iil) False (iv) True.

(i) Applying the method of separation of variahles techniques, tbe solution to tbe .D.E. $3u_x + 2u_y = 0$ is ...., where $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$.

Ans. $c_1 c_2 e^{hx/3} e^{-hy/2}$

Match the column for the items of the left side to tbat of right side :

(j) A second order P.D.E. in the function 'u' of two independent variahles x, y ven with usual symhols

$Au_{xx} + Bu_{xy} + Cu_{yy} + F(u) = 0$, then

(i) Hyperholic (a) $B^2 - 4AC = 0$ (ii) Parabolic (h) $B^2 - 4AC < 0$

(iii) Elllptic (c) $B^2 - 4AC > 0$ (iv) Not classified (d) $A = B = C = 0$

Ans. (i) (b), (ii) (a), (iii) (c) (iv) (d).

SECTION -B

Note : **Attempt any three questions from this section. all questions carry equal marks.** **$3 \times 10 = 30$**

2. (a) Apply the method of variation of parameters to solve the ordinary differential equations

$$\frac{d^2y}{dx^2} + y = \tan x, \quad 0 \quad x < \pi/2$$

Ans. Here $R = \tan x$

A.E. is $m^2 + 1 = 0 \Rightarrow m = \pm i$

C.F $= c_1 \cos x + c_2 \sin x$

Let $y_1 = \cos x,\ y_2 = \sin x$

$y_1' = -\sin x,\ y_2' = \cos x$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\text{P.I.} = -y_1 \int \frac{y_2 R}{w} dx + y_2 \int \frac{y_1 R}{w} dx$$

$$= -\cos x \int \frac{\sin x . \tan x}{1} dx + \sin x \int \frac{\cos x . \tan x}{1} dx = -\cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \sin x\, dx$$

$$= -\cos x \int \left(\frac{1 - \cos^2 x}{\cos x}\right) dx + \sin x(-\cos x) = -\cos x \int (\sec x - \cos x) dx - \sin x \cos x$$

$$= -\cos x\,[\log(\sec x + \tan x) - \sin x] - \sin x \cos x = -\cos x \log(\sec x + \tan x).$$

The complete solution is

$y = \text{C.F.} + \text{P.I.}$

$y = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x).$

Q. 2. (b) Show that the Bessel's function $J_n(x)$ is an even function when n is even and is odd function when n is odd. Express $J_6(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Ans. Suppose n is even

$$J_n(-x) = (-x)^n \sum_{m=0}^{\infty} \frac{(-1)^m (-x)^{2m}}{2^{2m+n} . m!(m+n)!}$$

for n even $(-1)^n = 1$ and $(-1)^{2m} = 1$

$$J_{-n}(-x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}\, m!(m+n)!} = J_n(x)$$

Thus $J_n(x)$ is an even function.
Suppose n is odd. Then $(-1)^n = -1$.

$$J_n(-x) = -x^n \sum_{m=0}^{\infty} \frac{(-1)^n x^{2m}}{2^{2m+n}\, m!(m+n)!} = -J_n(x).$$

This $J_n(x)$ is an odd function.
By the recurrence relation

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \qquad \text{....(1)}$$

Put $n = 1, 2, 3, 4, 5,$ in (1), we get

$$J_2 = \frac{2}{x} J_1(x) - J_0(x) \qquad \text{...(2)}$$

$$J_3 = \frac{4}{x} J_2(x) - J_1(x) \qquad \text{...(3)}$$

$$J_4 = \frac{6}{x} J_3(x) - J_2(x) \qquad \text{...(4)}$$

$$J_5 = \frac{8}{x} J_4(x) - J_3(x) \qquad \text{...(5)}$$

$$J_6 = \frac{10}{x} J_5(x) - J_4(x) \qquad \text{...(6)}$$

Substituting (3) in (2)

$$J_3 = \left(\frac{8}{x^2} - 1\right) J_1 - \frac{4}{x} J_0 \qquad \text{...(7)}$$

Substituting (7) and (2) in (4),

$$J_4 = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1 + \left(1 - \frac{24}{x^2}\right) J_0 \qquad \text{...(8)}$$

Substituting (8) and (3) in (5)

$$J_5 = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right) J_1 + \left(\frac{12}{x} - \frac{192}{x^3}\right) J_0 \qquad \text{...(9)}$$

Substituting (9) and (8) in (6), we get

$$J_6(x) = \frac{10}{x}\left[\left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right) J_1 + \left(\frac{12}{x} - \frac{192}{x^3}\right) J_0\right] - \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1 - \left(1 - \frac{24}{x^2}\right) J_0$$

$$J_6(x) = \left(\frac{3840}{x^4} - \frac{768}{x^3} - \frac{2}{x}\right) J_1(x) + \left(\frac{144}{x^2} - 1 - \frac{1920}{x^4}\right) J_0(x).$$

Q. 2. (c) Use convolution theorem to find the inverse of the function $\dfrac{1}{(s^2 + a^2)^2}$.

Ans. Let $f(s) = \dfrac{1}{(s^2 + a^2)}$, $\qquad g(s) = \dfrac{1}{(s^2 + a^2)}$

$$L^{-1}\{f(s)\} = \frac{1}{a} \sin at = F(t), \qquad L^{-1}\{g(s)\} = \frac{1}{a} \sin at = G(t)$$

By convolution theorem

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(x)\,G(t-x)\,dx$$

$$= \int_0^t \frac{1}{a}\sin ax\,\frac{1}{a}\sin(at-ax)\,dx = \frac{1}{2a^2}\int_0^t 2\sin ax\,\sin(at-ax)\,dx$$

$$= \frac{1}{2a^2}\int_0^t [\cos(ax-at+ax) - \cos(ax+at-ax)]\,dx = \frac{1}{2a^2}\int_0^t [\cos(2ax-at) - \cos at]\,dx$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2ax-at)}{2a} - x\cos at\right]_0^t = \frac{1}{2a^2}\left[\left(\frac{1}{2a}\sin at - t\cos at\right) - \left(-\frac{1}{2a}\sin at\right)\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{a}\sin at - t\cos at\right] = \frac{1}{2a^3}(\sin at - at\cos at)$$

Q. 2. (d) Obtain the Fourier series of $f(x) = \left(\dfrac{\pi - x}{2}\right)$ in the interval $(0, 2\pi)$ and hence deduce $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots\ldots$

Ans. $f(x) = \left(\dfrac{\pi - x}{2}\right)$

Fourier series of $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)dx = \frac{1}{2\pi}\left(\pi x - \frac{x^2}{2}\right)_0^{2\pi} = \frac{1}{2\pi}[(2\pi^2 - 2\pi^2) - (0-0)] = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\cos nx\,dx = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{(\pi - 2\pi)\left(\frac{\sin 2n\pi}{n}\right) - \frac{\cos 2n\pi}{n^2}\right\} - \left\{0 - \frac{1}{n^2}\right\}\right] = \frac{1}{2\pi}\left[-\frac{(-1)^{2n}}{n^2} + \frac{1}{n^2}\right] = \frac{1}{2\pi}\left[-\frac{1}{n^2} + \frac{1}{n^2}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\sin nx\,dx = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{-(\pi - 2\pi)\frac{\cos 2n\pi}{n} - 0\right\} - \left(\frac{-\pi}{n}\right)\right] = \frac{1}{2\pi}\left[\frac{\pi(-1)^{2n}}{n} + \frac{\pi}{n}\right] = \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right] = \frac{1}{n}.$$

Hence the required fourier series is

$$\left(\frac{\pi - x}{2}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\sin nx$$

$$\frac{\pi - x}{2} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \frac{\sin 5x}{5} + \ldots\ldots$$

put $x = \frac{\pi}{2}$

$$\frac{\pi}{4} = \sin\frac{\pi}{2} + \frac{\sin\pi}{2} + \frac{\sin\frac{3\pi}{2}}{3} + \frac{\sin 2\pi}{4} + \frac{\sin\frac{5\pi}{2}}{5} + \ldots\ldots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \ldots\ldots\ldots$$

(e) A string of Length 'L' is stretched and fastened to two fixed points, Find the solution of the wave equation $y_{tt} = a^2 y_{xx}$ when initial displacement is $y(x,0) = f(x) = b\sin\left(\frac{\pi x}{L}\right)$, where symbols have usual meaning.

Ans. Given wave equation $y_{tt} = a^2 y_{xx}$ ...(1)

Initial conditions $y(x,0) = f(x) = b\sin\left(\frac{\pi x}{L}\right)$ ...(2)

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \ldots(3)$$

Boundary conditions

$y(0,t) = 0$ ...(4)

$y(L,t) = 0$ ...(5)

Let $y = X(x)T(t)$ be the solution of eq^n (1)

$$(1) \Rightarrow \frac{T''}{a^2 T} = \frac{X''}{X} = \text{const.}$$

When const $= -p^2$

$\frac{X''}{X} = -p^2$	$\frac{T''}{a^2 T} = -p^2$
$X'' = -p^2 X$	$T'' = -p^2 a^2 T$
$X'' + p^2 X = 0$	$T'' + p^2 a^2 T = 0$
$m^2 + p^2 = 0 \Rightarrow m = \pm pi$	$m^2 + a^2 p^2 = 0 \Rightarrow m = \pm api$
$X = c_1 \cos px + c_2 \sin px$	$T = c_3 \cos apt + c_4 \sin apt$

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos apt + c_4\, a \sin pt) \quad \ldots(6)$$

using B.C. (4), $y(0,t) = 0$ in (6), we get

$$0 = c_1(c_3 \cos apt + c_4 \sin apt)$$

$$\Rightarrow \boxed{c_1 = 0}$$

using B.C. (5), $y(L,t) = 0$ in (6) we get

$$0 = (c_1 \cos as.\, pL + c_2 \sin pL)(c_3 \cos apt + c_4 \sin apt)$$

$$\Rightarrow c_2 \sin pL = 0 \Rightarrow \sin pL = 0 = \sin n\pi$$

$$\boxed{p = \frac{n\pi}{L}}$$

$$(6) \Rightarrow \quad y(x,t) = c_2 \sin\frac{n\pi x}{L}\left(c_3 \cos\frac{n\pi at}{L} + c_4 \sin\frac{n\pi at}{L}\right) \quad \ldots(7)$$

$$\frac{\partial y}{\partial t} = c_2 \sin \frac{n\pi x}{L}\left(-c_3 \sin \frac{n\pi at}{L} + c_4 \cos \frac{n\pi at}{L}\right)\left(\frac{n\pi a}{L}\right)$$

using I.C. (3), $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

$$0 = c_2 \sin \frac{n\pi x}{L}(-0 + c_4)\left(\frac{n\pi a}{L}\right)$$

$\Rightarrow \boxed{c_4 = 0}$

$$(7) \Rightarrow y(x,t) = c_2 c_3 \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right) \quad ...(8)$$

using I.C. (2), $y(x, 0) = f(x) = b \sin\left(\frac{\pi x}{L}\right)$ in (8), we get

$$b \sin\left(\frac{\pi x}{L}\right) = c_2 c_3 \sin\left(\frac{n\pi x}{L}\right)$$

$\Rightarrow c_2 c_3 = b$ and $n = 1$.

Hence the required solution is

$$y(x,t) = b \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi at}{L}\right)$$

SECTION - C

Note : **Attempt all questions from this section, selecting any two parts from each question. All questions carry equal marks.** **[5 × 2] × 5 = 50**

Q. 3. (a) The equations of motion of a particle are given by $\frac{dx}{dt} + wy = 0$, $\frac{dy}{dt} - wx = 0$ Find the path of the particle and show that it is a circle.

Ans. The equation of motion

$$\frac{dx}{dt} + wy = 0$$

$$\frac{dy}{dt} - wx = 0$$

or $Dx + wy = 0$ ...(1)

$wx + Dy = 0$ ...(2)

Eliminating y in (1) & (2), by multiplying (1) by D and (2) by w and subtracting, we get

$$(D^2 + w^2)x = 0$$

A.E. is $m^2 + w^2 = 0 \Rightarrow m = \pm iw$

$$x = c_1 \cos wt + c_2 \sin wt \quad ...(3)$$

$$\frac{dx}{dt} = -c_1 w \sin wt + c_2 w \cos wt.$$

from (1), $y = -\frac{1}{w}(Dx) = -\frac{1}{w}(-c_1 w \sin wt + c_2 w \cos wt)$

$$y = c_1 \sin wt - c_2 \cos wt \quad ...(4)$$

squaring and adding (3) & (4), we get

$x^2 + y^2 = c_1^2 + c_2^2$ = constant

Hence the path of the particle is a circle.

Q. 3. (b) Integrate the differential equation

$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + w^2x = a\cos pt$ and give the physical interpretation of the complete solution. Also deduce that the solution takes the form

$$x = \frac{a\cos(pt-\theta)}{\sqrt{(w^2-p^2)^2+4k^2p^2}}\left(\tan\theta = \frac{2kp}{w^2-p^2}\right) \text{As } t \to \infty$$

Ans. $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + w^2x = a\cos pt$

A E. is $m^2 + 2km + w^2 = 0$

$m = -k \pm \sqrt{k^2 - w^2}$

$$\text{C.F.} = e^{-kt}\left[c_1e^{t\sqrt{k^2-w^2}} + c_2e^{-t\sqrt{k^2-w^2}}\right]$$

It represents the free oscillations of the system which die out as $t \to \infty$

$$\text{P.I.} = \frac{1}{D^2+2kD+w^2}a\cos pt = a\frac{1}{-p^2+2kD+w^2}\cos pt$$

$$= a.\frac{[(w^2-p^2)-2kD]}{(w^2-p^2)^2-4k^2D^2}\cos pt = a\frac{[(w^2-p^2)-2kD]}{(w^2-p^2)^2+4k^2p^2}\cos pt$$

$$= \frac{a}{(w^2-p^2)^2+4k^2p^2}[(w^2-p^2)\cos pt + 2kp\sin pt]$$

putting $R\cos\theta = (w^2-p^2)$, $R\sin\theta = 2kp$

$R = \sqrt{(w^2-p^2)^2+4k^2p^2}$, $\tan\theta = \frac{2kp}{w^2-p^2}$

$$= \frac{a}{(w^2-p^2)^2+4k^2p^2}R\cos(pt-\theta) = \frac{a\sqrt{(w^2-p^2)^2+4k^2p^2}}{(w^2-p^2)^2+4k^2p^2}\cos(pt-\theta)$$

$$\text{P.I.} = \frac{a\cos(pt-\theta)}{\sqrt{(w^2-p^2)^2+4k^2p^2}}, \text{ where } \tan\theta = \frac{2kp}{w^2-p^2}$$

Which represents the forced oscillation of the system having

(i) constant amplitude $= \frac{a}{\sqrt{(w^2-p^2)^2+4k^2p^2}}$

(ii) period $= \frac{2\pi}{p}$

The complete solution is

x = C.F. + P.I.

$$x = e^{-kt}\left[c_1e^{t\sqrt{k^2-w^2}} + c_2e^{-t\sqrt{k^2-w^2}}\right] + \frac{a\cos(pt-\theta)}{\sqrt{(w^2-p^2)^2+4k^2p^2}}$$

when $t \to \infty$, the free oscillation (C.F.) die away while the forced oscillations continue giving the steady state motion.Then

$$x = \frac{a\cos(pt-\theta)}{\sqrt{(w^2-p^2)^2+4k^2p^2}} \text{ where } \tan\theta = \frac{2kp}{w^2-p^2}$$

Q. 3. (c) Find the complete solution of the differential equation $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \cos x$.

Ans. A.E. is $m^2 - 2m + 1 = 0, \qquad (m-1)^2 = 0 \Rightarrow m = 1, 1$

$$\text{C.F.} = (c_1 + c_2x)\,e^x$$

$$\text{P.I} = \frac{1}{D^2-2D+1}e^x x\cos x = \frac{1}{(D-1)^2}e^x(x\cos x) = e^x\frac{1}{(D+1-1)^2}x\cos x$$

$$= e^x\frac{1}{D^2}x\cos x = e^x\frac{1}{D}\int x\cos x\,dx = e^x\frac{1}{D}[x(\sin x) - (-\cos x)] = e^x\int(x\sin x + \cos x)\,dx$$

$$= e^x[x(-\cos x) - (-\sin x) + \sin x] = -e^x[-x\cos x + 2\sin x] = e^x(x\cos x - 2\sin x)$$

The complete solution is

$y = \text{C.F.} + \text{P.I.},\ y = (c_1 + c_2x)e^x + e^x(x\cos x - 2\sin x)$

Q. 4 (a) For the Bessel's function, prove that

$$J_{\frac{1}{2}}(x) = \left(\sqrt{\frac{2}{\pi x}}\right)\sin x.$$

Ans. $J_n(x) = \displaystyle\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$$J_n(x) = \left(\frac{x}{2}\right)^n\left[\frac{1}{\Gamma(n+1)} - \frac{1}{1!\,\Gamma(n+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^4 \ldots\ldots\right]$$

put $n = \dfrac{1}{2}$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\Gamma\frac{3}{2}} - \frac{1}{1!\,\Gamma\frac{5}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\frac{7}{2}}\left(\frac{x}{2}\right)^4 \ldots\ldots\ldots\right]$$

$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\Gamma\frac{1}{2}} - \frac{1}{\frac{3}{2}.\frac{1}{2}.\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^4\ldots\ldots\right] = \frac{\sqrt{x}}{\sqrt{2}\,\Gamma\frac{1}{2}}\left[\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!}\ldots\ldots\right]$$

Now multiplying the series by $x/2$ and outside by $\dfrac{2}{x}$, we get

$$J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}\ldots\ldots\ldots\right] = \sqrt{\frac{2}{\pi x}}\sin x$$

Q. 4 (b) Estimate the recurrence relation for Legendre's polynomials :

$$\boxed{x P'_n(x) = n P_n(x) + P'_{n-1}(x)}$$

Ans. $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$ ...(1)

Differentiating (1) partially w.r.t x, we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum_{n=0}^{\infty} P_n'(x)t^n, \quad t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n'(x)t^n \qquad ...(2)$$

Again differentiating (1) partially w.r to t, we get

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} n P_n(x).t^{n-1} \qquad ...(3)$$

Dividing (3) by (2), we get

$$\frac{x-t}{t} = \frac{\Sigma n P_n(x) t^{n-1}}{\Sigma P_n'(x) t^n}$$

$$\Sigma n P_n(x) t^n = (x-t)\, \Sigma P_n'(x) t^n$$

Equating coefficients of t^n from both sides, we get

$$n P_n(x) = x P_n'(x) - P_{n-1}'(x)$$

or $x P_n'(x) = n P_n(x) + P_{n-1}'(x)$

Q. 4. (c) Express the polynomial $f(x) = 4x^3 - 2x^2 - 3x + 8$ in terms of Legendre polynomials.

Ans. $f(x) = 4x^3 - 2x^2 - 3x + 8 = aP_3(x) + bP_2(x) + cP_1(x) + dP_0(x)$

$$4x^3 - 2x^2 - 3x + 8 = a\left(\frac{5x^3-3x}{2}\right) + b\left(\frac{3x^2-1}{2}\right) + c(x) + d(1)$$

On comparing the coefficient of various power of x.

$$\frac{5a}{2} = 4 \Rightarrow a = \frac{8}{5}$$

$$\frac{3b}{2} = -2 \Rightarrow b = -\frac{4}{3}$$

$$-\frac{3a}{2} + c = -3 \Rightarrow c = -3 + \frac{3a}{2} = -3 + \frac{3}{2}\left(\frac{8}{5}\right) = \frac{-3}{5}$$

$$-\frac{b}{2} + d = 8 \Rightarrow d = 8 + \frac{b}{2} = 8 + \frac{1}{2}\left(\frac{-4}{3}\right) = \frac{22}{3}$$

Hence

$$f(x) = \frac{8}{5}P_3(x) - \frac{4}{3}P_2(x) - \frac{3}{5}P_1(x) + \frac{22}{3}P_0(x).$$

Q. 5. (a) Using Laplace transformation, show that $\int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$.

Ans. Let $f(t) = \sin t$, $L\{f(t)\} = L(\sin t) = \frac{1}{s^2+1} = F(s)$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds,\ L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \left(\frac{1}{s^2+1}\right)ds = [\tan^{-1} s]_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty e^{-0t}\left(\frac{\sin t}{t}\right)dt = L\left(\frac{\sin t}{t}\right),\ \text{when } s = 0 = \left[\frac{\pi}{2} - \tan^{-1} s\right]_{s=0} = \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2}.$$

Q. 5. (b) A particle moves in a line so that its displacement x from a fixed point at any time t, is given by $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 80 \sin 5t$. Using Laplace transform, find i displacement at any time t if initially particle is at rest at $x = 0$.

Ans. $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 80 \sin 5t$ ...(

$x(0) = 0, x'(0) = 0$

Taking Laplace transform of (1) we get

$[s^2L(x) - sx(0) - x'(0)] + 4[sL(x) - x(0)] + 5L(x) = 80L(\sin 5t)$

$(s^2 + 4s + 5)L(x) = 80\left(\frac{5}{s^2+25}\right),\ L(x) = 400\left[\frac{1}{(s^2+25)(s^2+4s+5)}\right]$

on inversion,

$$x = 400\,L^{-1}\left[\frac{1}{(s^2+25)(s^2+4s+5)}\right]$$

$\frac{1}{(s^2+25)(s^2+4s+5)} = \frac{As+B}{s^2+25} + \frac{Cs+D}{s^2+4s+5}$

$1 = (As+B)(s^2+4s+5) + (Cs+D)(S^2+25)$

$A + C = 0 \Rightarrow C = -A$

$B + D + 4A = 0$

$4B + 5A + 25C = 0$

$5B + 25D = 1$

on solving, we get

$B = -\frac{1}{40}, D = \frac{9}{200}, A = -\frac{1}{8}, C = \frac{1}{8}$

$\frac{1}{(s^2+25)(s^2+4s+5)} = \frac{\left(-\frac{1}{8}s - \frac{1}{40}\right)}{s^2+25} + \frac{\left(\frac{1}{8}s + \frac{9}{200}\right)}{s^2+4s+5}$

$$x = 400L^{-1}\left[-\frac{1}{40}\left(\frac{5s+1}{s^2+25}\right) + \frac{1}{200}\left(\frac{25s+9}{(s^2+4s+5)}\right)\right]$$

$$= \frac{400}{200}\left[-5L^{-1}\left(\frac{5.s}{s^2+25}\right) - 5\,L^{-1}\left(\frac{1}{s^2+25}\right) + L^{-1}\left(\frac{25s+9}{s^2+4s+5}\right)\right]$$

$$= -50\cos 5t - \frac{10}{5}\sin 5t + 2L^{-1}\left[\frac{25(s+2) - 50 + 9}{(s+2)^2+1}\right]$$

$$= -50\cos 5t - 2\sin 5t + 2e^{-2t} L^{-1}\left[\frac{25s-41}{s^2+1}\right]$$

$$x = -50\cos 5t - 2\sin 5t + 2e^{-2t}(25\cos t - 41\sin t)$$

Q. 5. (c) Find the Laplace transform of the function

$$f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$$

Ans. $f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$

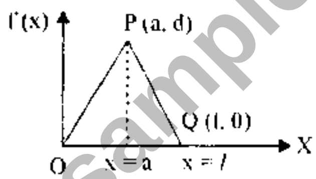
$$f(t) = (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$
$$f(t) = (t-1)u(t-1) - (t-1-3+t)u(t-2) + (t-3)u(t-3)$$
$$f(t) = (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3).$$

Taking Laplace transform, we get,

$$L\{f(t)\} = L\{(t-1)u(t-1)\} - 2L\{(t-2)u(t-2)\} + L\{(t-3)u(t-3)\}$$

$$= e^{-s}L(t) - 2e^{-2s}L(t) + e^{-3s}L(t) = \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$

Q. 6. (a) Find the half period sine series for $f(x)$ given in the range $(0, l)$ by the graph OPQ as shown in figure.



Ans. $f(x) = \begin{cases} \dfrac{d}{a}x, & 0<x<a \\ \dfrac{d}{a-l}(x-l), & a<x<l \end{cases}$

Half period sine series $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ ...(1)

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\left[\int_0^a\left(\frac{d}{a}x\right)\sin\frac{n\pi x}{l}dx + \int_a^l \frac{d}{a-l}(x-l)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2}{l}\left[\frac{d}{a}\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right\}_0^a + \frac{d}{(a-l)}\left\{(x-l)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right\}_a^l\right]$$

$$= \frac{2}{l}\left[\frac{d}{a}\left\{\frac{-al}{n\pi}\cos\frac{n\pi a}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi a}{l}\right\} - \frac{d}{a-l}\left\{-\frac{(a-l)l}{n\pi}\cos\frac{n\pi a}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi a}{l}\right\}\right]$$

$$= \frac{2}{l}\left[-\frac{dl}{n\pi}\cos\frac{n\pi a}{l} + \frac{dl^2}{an^2\pi^2}\sin\frac{n\pi a}{l} + \frac{dl}{n\pi}\cos\frac{n\pi a}{l} - \frac{dl^2}{(a-l)n^2\pi^2}\sin\frac{n\pi a}{l}\right]$$

$$= \frac{2}{l}\frac{dl^2}{n^2\pi^2}\left(\frac{1}{a} - \frac{1}{a-l}\right)\sin\frac{n\pi a}{l} = \frac{2dl}{n^2\pi^2}\left(\frac{a-l-a}{a(a-l)}\right)\sin\frac{n\pi a}{l}$$

$$b_n = -\frac{2dl^2}{n^2\pi^2 a(a-l)} \sin\frac{n\pi a}{l} = \frac{2dl^2}{n^2\pi^2 a(l-a)} \sin\frac{n\pi a}{l}$$

Hence from (1),

$$f(x) = \sum_{n=1}^{\infty} \frac{2dl^2}{n^2\pi^2 a(l-a)} \sin\frac{n\pi a}{l} \sin\frac{n\pi x}{l}$$

Q. 6. (b) Find the Fourier series expansion for $f(x) = x + \frac{x^2}{4}$, $-\pi \le x \le \pi$.

Ans. $f(x) = x + \frac{x^2}{4}, -\pi \le x \le \pi$

Fourier series for function $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + \frac{x^2}{4}\right)dx = \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{12}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} + \frac{\pi^3}{12}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{12}\right)\right] = \frac{1}{\pi}\left(\frac{2\pi^3}{12}\right) = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + \frac{x^2}{4}\right)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\left(x + \frac{x^2}{4}\right)\left(\frac{\sin nx}{n}\right) - \left(1 + \frac{2x}{4}\right)\left(-\frac{\cos nx}{n^2}\right) + \left(\frac{2}{4}\right)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(1 + \frac{2\pi}{4}\right)\frac{\cos. n\pi}{n^2} - \left(1 - \frac{2\pi}{4}\right)\frac{\cos n\pi}{n^2}\right] = \frac{1}{\pi}\left[1 + \frac{2\pi}{4} - 1 + \frac{2\pi}{4}\right]\frac{\cos n\pi}{n^2} = \frac{\cos n\pi}{n^2} = \frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + \frac{x^2}{4}\right)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\left(x + \frac{x^2}{4}\right)\left(-\frac{\cos nx}{n}\right) - \left(1 + \frac{2x}{4}\right)\left(-\frac{\sin nx}{n^2}\right) + \left(\frac{2}{4}\right)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{-\left(\pi + \frac{\pi^2}{4}\right)\frac{\cos n\pi}{n} + \frac{1}{2}\frac{\cos n\pi}{n^3}\right\} - \left\{-\left(-\pi + \frac{\pi^2}{4}\right)\frac{\cos n\pi}{n} + \frac{1}{2}\frac{\cos n\pi}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[\left(-\pi - \frac{\pi^2}{4}\right)\frac{\cos n\pi}{n} + \frac{1}{2}\frac{\cos n\pi}{n^3} + \left(-\pi + \frac{\pi^2}{4}\right)\frac{\cos n\pi}{n} - \frac{1}{2}\frac{\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\left(-\pi - \frac{\pi^2}{4} - \pi + \frac{\pi^2}{4}\right)\frac{\cos n\pi}{n}\right]$$

$$b_n = \frac{1}{\pi}(-2\pi)\frac{\cos n\pi}{n}, \quad b_n = -\frac{2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2} + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

Q. 6. (c) Solve the P.D.E

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$$

Ans. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$

A.E. is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

C.F. $= f_1(y+x) + xf_2(y+x)$

$$\text{P.I} = \frac{1}{D^2 - 2DD' + D'^2}\sin x$$

$$D^2 = -1,\ DD' = 0,\ D'^2 = 0, \quad = \frac{1}{-1-0+0}\sin x = -\sin x$$

The complete solution is $y =$ C.F. + P.I.

$y = f_1(y+x) + xf_2(y+x) - \sin x$

Q. 7. (a) Find tbe temperature in a bar of lengtb 2 whose end are kept at zero and eral surface is insulated if the initial temperature is $\sin\frac{\pi x}{2} + 3\sin 5\pi\frac{x}{2}$.

Ans. $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ ...(1)

Boundary condition are

$u(0,t) = 0,\ u(2,t) = 0$

Initial condition is $u(x,0) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$

Let $u = X(x)\,T(t)$ be the solution of equation(1)

$$(1) \Rightarrow \frac{T'}{c^2 T} = \frac{X''}{X} = \text{constant} = -p^2$$

$\frac{X''}{X} = -p^2$	$\frac{T'}{c^2 T} = -p^2$
$X'' + p^2 X = 0$	$\frac{T'}{T} = -c^2 p^2$
$m^2 + p^2 = 0$	$\log T = -c^2p^2t + \log c_3$
$m = \pm pi$	$T = c_3 e^{-c^2p^2t}$

$X = c_1\cos px + c_2\sin px$

$u(x,t) = (c_1\cos px + c_2\sin px)\,c_3 e^{-c^2p^2t}$ ...(2)

using B. C. $u(0,t) = 0$

$0 = c_1 c_3 e^{-c^2p^2t} \Rightarrow c_1 = 0$ ...(3)

Using B. C. $u(2,t) = 0$

$0 = (c_1\cos 2p + c_2\sin 2p)\,c_3 e^{-c^2p^2t}$

$c_2\sin 2p = 0 \Rightarrow \sin 2p = 0 = \sin n\pi$

$p = \frac{n\pi}{2}$ ...(4)

putting (3) & (4) in (2), we get

$$u(x,t) = c_2 c_3 \sin\frac{n\pi x}{2} e^{-\frac{n^2\pi^2c^2t}{4}}$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} e^{-\frac{n^2\pi^2c^2t}{4}} \quad ...(5)$$

using I.C in (5)

$$u(x,0) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$$

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$\Rightarrow b_1 = 1$, $n = 1$ and $b_5 = 3$, $n = 5$

Hence the required solution in

$$u(x,t) = \sin\frac{\pi x}{2} e^{-\frac{\pi^2c^2t}{4}} + 3\sin\frac{5\pi x}{2} e^{-\frac{25\pi^2c^2t}{4}}$$

Q. 7. (b) Apply the metbod of separation of variables to solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

Ans. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

Let $Z = X(x)\, Y(y)$ ...(1)

Where X is a function of x alone and Y that of y alone. Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0, \text{ where } X' = \frac{dX}{dx}, Y' = \frac{dY}{dy}$$

separating the variables, we get

$$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = \text{constant} = k \quad ...(2)$$

$$\therefore \frac{X'' - 2X'}{X} = k \Rightarrow X'' - 2X' - kX = 0 \quad ...(3)$$

$$\text{and } \frac{-Y'}{Y} = k \Rightarrow Y' + kY = 0 \quad ...(4)$$

To solve equation (3), the auxiliary equation is

$$m^2 - 2m - k = 0 \Rightarrow m = 1 \pm \sqrt{1+k}$$

$$X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$$

and the solution of (4) is $Y = c_3 e^{-ky}$

from (1), we get

$$z = \left[c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}\right] c_3 e^{-ky}, \quad z = [A\, e^{(1+\sqrt{1+k})x} + B\, e^{(1-\sqrt{1+k})}x]\, e^{-ky}$$

Q. 7. (c) Solve tbe P.D.E by separation of variables method, $u_{xx} = u_y + 2u$, $u(0, y) = 0$, $\frac{\partial}{\partial x} u(0, y) = 1 + e^{-3y}$

Ans. $u_{xx} = u_y + 2u$ ...(1)

$u(0, y) = 0, \frac{\partial}{\partial x} u(0, y) = 1 + e^{-3y}$

Let $u(x, y) = X(x)\, Y(y)$ ...(2)

Substituting in the given equation (1), we have

$X''Y = XY' + 2XY$

$\frac{X''}{X} - 2 = \frac{Y'}{Y} = k$

$X'' - 2X = kX \Rightarrow X'' - (2 + k)X = 0$

$m^2 - (2 + k) = 0$

$m = \pm\sqrt{2 + k}$

$X = c_1 e^{x\sqrt{2+k}} + c_2 e^{-x\sqrt{2+k}}$

$\frac{Y'}{Y} = k \Rightarrow \log Y = ky + \log c_3$

$Y = c_3 e^{ky}$

$u(x, y) = \left[c_1 e^{x\sqrt{2+k}} + c_2 e^{-x\sqrt{2+k}}\right] c_3 e^{ky}$

$u(x, y) = (A\, e^{x\sqrt{2+k}} + B\, e^{-x\sqrt{2+k}}) e^{ky}$ ....(3)

$\frac{\partial u}{\partial x} = \left[A e^{x\sqrt{2+k}} - B e^{-x\sqrt{2+k}}\right] \sqrt{2+k}\, e^{ky}$

using in equation (3), $u(0, y) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$

$\frac{\partial u}{\partial x}(0, y) = 1 + e^{-3y} \Rightarrow [A - B]\sqrt{2+k}\, e^{ky} = 1 + e^{-3y}$

$(A + B)(\sqrt{2+k}) e^{ky} = 1 + e^{-3y}$

$2A(\sqrt{2+k})\, e^{ky} = 1 + e^{-3y}$

$\Rightarrow \quad 2A\sqrt{2+k} = 1,\ k = 0$

and $2A\sqrt{2+k} = 1,\ k = -3$

When $k = 0$, $A = \frac{1}{2\sqrt{2+0}} = \frac{1}{2\sqrt{2}}$ and $B = -\frac{1}{2\sqrt{2}}$

When $k = -3$, $A = \frac{1}{2\sqrt{2-3}} = \frac{1}{2i}$ and $B = \frac{1}{2i}$

$u(x, y) = \frac{1}{2\sqrt{2}}\left(e^{x\sqrt{2}} - e^{-x\sqrt{2}}\right) + \left[\frac{e^{ix} - e^{-ix}}{2i}\right] e^{-3y}$

$= \frac{2\sin h(x\sqrt{2})}{2\sqrt{2}} + \sin x . e^{-3y}$

$u(x, y) = \frac{1}{\sqrt{2}} \sin h(x\sqrt{2}) + e^{-3y} \sin x$