

B.Tech.

SECOND SEMESTER EXAMINATION, 2007-08

MATHEMATICS-II

(TAS-204)

Time : 3 Hours]

[Total Marks : 100

Note : (1) Attempt all questions.

(2) All questions carry equal marks.

(3) In case of numerical problems assume data wherever not provided.

(4) Be precise in your answer.

Q. 1. Attempt any four parts of the following :

5 × 4 = 20

(a) Solve $(y + \sqrt{x^2 + y^2})dx - xdy = 0,$

$y(1) = 0.$

Ans.

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \quad \dots(1)$$

This is a homogenous equation, so

Put $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

from (1) \Rightarrow

$$v + x \frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$$

$$x \frac{dv}{dx} = \frac{v + \sqrt{1 + v^2}}{1} - v$$

$$\frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

$$\log(v + \sqrt{1 + v^2}) = \log x + \log c$$

$$v + \sqrt{1 + v^2} = cx$$

$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = cx$$

$$y + \sqrt{x^2 + y^2} = cx^2 \quad \dots(2)$$

put $x = 1, y = 0,$ in eqn (2) $\Rightarrow 0 + \sqrt{1 + 0} = c$

$c = 1$

The required solution is

$$y + \sqrt{x^2 + y^2} = x^2$$

Q. 1. (h) Solve the following differential equation :

$$(\cos x - x \cos y)dy - (\sin y + y \sin x)dx = 0$$

Ans.

Comparing with $Mdx + Ndy = 0$

we get $M = -(\sin y + y \sin x),$

$N = \cos x - x \cos y$

$$\frac{\partial M}{\partial y} = -\cos y - \sin x, \quad \frac{\partial N}{\partial x} = -\sin x - \cos y$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given equation is exact.

On Integrating, we get

$$-\int (\sin y + y \sin x)dx = c$$

$$-[x \sin y + y(-\cos x)] = c$$

$$y \cos x - x \sin y = c.$$

Q. 1. (c) Find the solution of following differential equation :

$$(D^2 - 4D - 5)y = e^{2x} + 3 \cos(4x + 3)$$

where $D = \frac{d}{dx}$

Ans.

A.E. is $m^2 - 4m - 5 = 0$

$$m^2 - 5m + m - 5 = 0$$

$$m = -1, 5$$

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{5x}$$

$$\text{P.I.} = \frac{1}{D^2 - 4D - 5} [e^{2x} + 3 \cos(4x + 3)]$$

$$= \frac{1}{D^2 - 4D - 5} e^{2x}$$

$$+ 3 \frac{1}{D^2 - 4D - 5} \cos(4x + 3)$$

$$= \frac{1}{4 - 8 - 5} e^{2x}$$

$$+ 3 \frac{1}{-18 - 4D - 5} \cos(4x + 3)$$

$$= -\frac{1}{9} e^{2x} + 3 \frac{1}{-4D - 21} \cos(4x + 3)$$

$$= -\frac{1}{9} e^{2x} - 3 \frac{1}{4D + 21} \cos(4x + 3)$$

$$= -\frac{1}{9} e^{2x} - 3 \frac{(4D - 21)}{16D^2 - 441} \cos(4x + 3)$$

$$= -\frac{1}{9} e^{2x} - 3 \frac{(4D - 21)}{-256 - 441} \cos(4x + 3)$$

$$= -\frac{1}{9} e^{2x} + \frac{3}{697} [-16 \sin(4x + 3)$$

$$- 21 \cos(4x + 3)]$$

$$= -\frac{1}{9} e^{2x} - \frac{48}{697} \sin(4x + 3)$$

$$- \frac{63}{697} \cos(4x + 3)$$

The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 e^{2x} + c_2 e^{5x} - \frac{e^{2x}}{9} - \frac{48}{897} \sin(4x + 3)$$

$$- \frac{83}{697} \cos(4x + 3)$$

Q. 1. (d) Solve the following simultaneous differential equations:

$$\frac{dx}{dt} + 5x - 2y = t$$

$$\frac{dy}{dt} + 2x + y = 0$$

Also show that $x = y = 0$ when $t = 0$

Ans.

$$\Rightarrow (D + 5)x - 2y = t \quad \dots(1)$$

$$2x + (D + 1)y = 0 \quad \dots(2)$$

Multiplying (1) by $(D + 1)$ and (2) by 2 and

adding, we get

$$[(D + 5)(D + 1) + 4]x = (D + 1)t$$

$$(D^2 + 6D + 9)x = 1 + t$$

$$\text{A.E. is } m^2 + 6m + 9 = 0$$

$$(m + 3)^2 = 0$$

$$m = -3, -3$$

$$\text{C.F.} = (c_1 + c_2 t)e^{-3t}$$

$$\text{P.I.} = \frac{1}{D^2 + 6D + 9} (1 + t)$$

$$= \frac{1}{9} \left[1 + \left(\frac{2D}{3} + \frac{D^2}{9} \right) \right]^{-1} (1 + t)$$

$$= \frac{1}{9} \left[1 - \frac{2}{3}D - \frac{D^2}{9} \right] (1 + t)$$

$$= \frac{1}{9} \left[(1 + t) - \frac{2}{3} \right] = \frac{1}{9} \left(t + \frac{1}{3} \right)$$

$$x = \text{C.F.} + \text{P.I.}$$

$$x = (c_1 + c_2 t)e^{-3t} + \frac{t}{9} + \frac{1}{27}$$

$$\frac{dx}{dt} = (c_1 + c_2 t)(-3e^{-3t}) + (c_2)e^{-3t} + \frac{1}{9}$$

From (1) \Rightarrow

$$2y = \frac{dx}{dt} + 5x - t$$

$$= -3(c_1 + c_2 t)e^{-3t} + c_2 e^{-3t} + \frac{1}{9}$$

$$+ 5 \left[(c_1 + c_2 t)e^{-3t} + \frac{t}{9} + \frac{1}{27} \right] - t$$

$$= (-3c_1 - 3c_2 t + c_2 + 5c_1 + 5c_2 t)e^{-3t}$$

$$- \frac{4}{9}t + \frac{8}{27}$$

$$2y = (2c_1 + c_2 + 2c_2 t)e^{-3t} - \frac{4}{9}t + \frac{8}{27}$$

$$y = \left(c_1 + \frac{c_2}{2} + c_2 t \right) e^{-3t} - \frac{2}{9}t + \frac{4}{27}$$

The complete solution is

$$x = (c_1 + c_2 t) e^{-3t} + \frac{t}{9} + \frac{1}{27}$$

$$y = \left(c_1 + \frac{c_2}{2} + c_2 t \right) e^{-3t} + \frac{2}{9}t + \frac{4}{27}$$

Also given $x = y = 0$ when $t = 0$

$$0 = c_1 + \frac{1}{27} \Rightarrow c_1 = -\frac{1}{27}$$

$$0 = c_1 + \frac{c_2}{2} + \frac{4}{27} \Rightarrow c_2 = -\frac{2}{9}$$

The required solution is

$$x = -\frac{1}{27}(1 + 6t)e^{-3t} + \frac{t}{9} + \frac{1}{27}$$

$$y = -\frac{1}{27}(4 + 3t)e^{-3t} - \frac{2}{9}t + \frac{4}{27}$$

Q. 1. (e) Solve by method of variation of parameters

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = e^{-x} \log x.$$

Ans.

$$\text{Here } R = e^{-x} \log x$$

$$\text{A.E. is } m^2 + 2m + 1 = 0$$

$$(m + 1)^2 = 0$$

$$m = -1, -1$$

$$\text{C.F.} = (c_1 + c_2 x) e^{-x} = c_1 e^{-x} + c_2 x e^{-x}$$

$$\text{Let } y_1 = e^{-x}, y_2 = x e^{-x}$$

$$y_1' = -e^{-x}, y_2' = e^{-x} - x e^{-x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix}$$

$$= e^{-x} (e^{-x} - x e^{-x}) + x e^{-2x}$$

$$= -e^{-2x} - x e^{-2x} + x e^{-2x}$$

$$= -e^{-2x}$$

$$\text{P.I.} = -y_1 \int \frac{y_2 R}{W} dx + y_2 \int \frac{y_1 R}{W} dx$$

$$= -e^{-x} \int \frac{x e^{-x} e^{-x} \log x}{e^{-2x}} dx$$

$$+ x e^{-x} \int \frac{e^{-x} e^{-x} \log x}{e^{-2x}} dx$$

$$= -e^{-x} \int x \log x dx + x e^{-x} \int \log x \cdot 1 dx$$

$$= -e^{-x} \left(\log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right)$$

$$+ x e^{-x} \left(\log x \cdot x - \int \frac{1}{x} \cdot x dx \right)$$

$$= -e^{-x} \left(\frac{x^2}{2} \log x - \frac{x^2}{4} \right) + x e^{-x} (x \log x - x)$$

$$= -\frac{x^2 e^{-x}}{2} \log x + \frac{x^2 e^{-x}}{4}$$

$$+ x^2 e^{-x} \log x - x^2 e^{-x}$$

$$= \frac{x^2 e^{-x}}{2} \log x - \frac{3}{4} x^2 e^{-x}$$

The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = (c_1 + c_2 x) e^{-x} + \frac{x^2 e^{-x}}{2} \log x - \frac{3}{4} x^2 e^{-x}$$

Q. 1. (f) An inductance (L) of 2.0 H and a resistance (R) of 20 Ω are connected in series with an e.m.f. E volt. If the current (i) is zero, when $t = 0$, find the current (i) at the end of 0.01 second if $E = 100$ V, using the following differential equation :

$$L \frac{di}{dt} + iR = E$$

$$\text{Ans. } L \frac{di}{dt} + iR = E$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

...(1)

$$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{Rt/L}$$

The solution of (1) is

$$i(I.F.) = \int \frac{E}{L}(I.F.)dt + c$$

$$ie^{Rt/L} = \int \frac{E}{L}e^{Rt/L}dt + c$$

$$ie^{Rt/L} = \frac{E}{L} \cdot \frac{L}{R} e^{Rt/L} + c$$

$$i = \frac{E}{R} + ce^{-Rt/L} \quad \dots(2)$$

Initially, when $t = 0, i = 0$ so that

$$c = \frac{-E}{R}$$

$$i = \frac{E}{R}(1 - e^{-Rt/L}) \quad \dots(3)$$

when $t = 0.01, L = 20 \text{ H}, R = 20 \Omega,$
 $E = 100 \text{ V}.$

$$i = \frac{100}{20} \left(1 - e^{-\frac{20}{2} \times (0.01)} \right)$$

$$i = 5(1 - e^{-0.1})$$

$$i = 0.4758 \text{ amp}$$

Q. 2. Attempt any four parts of the following: $5 \times 4 = 20$

(a) Using Laplace transform, evaluate

$$\int_0^{\infty} e^{-t} \frac{\sin^2 t}{t} dt.$$

$$\text{Ans. Let } I = \int_0^{\infty} e^{-t} \frac{\sin^2 t}{t} dt$$

$$= \int_0^{\infty} e^{-st} \frac{\sin^2 t}{t} dt \text{ when } s = 1$$

$$= L \left\{ \frac{\sin^2 t}{t} \right\}, \text{ when } s = 1$$

....(i)

$$L \{ \sin^2 t \} = L \left\{ \frac{1 - \cos 2t}{2} \right\}$$

$$= \frac{1}{2} L \{ 1 \} - \frac{1}{2} L \{ \cos 2t \}$$

$$= \frac{1}{2s} - \frac{1}{2} \cdot \frac{s}{s^2 + 4} = f(s)$$

$$L \left\{ \frac{\sin^2 t}{t} \right\} = \int_s^{\infty} f(s) ds$$

$$= \int_s^{\infty} \left(\frac{1}{2s} - \frac{s}{2(s^2 + 4)} \right) ds$$

$$= \left[\frac{1}{2} \log s - \frac{1}{4} \log (s^2 + 4) \right]_s^{\infty}$$

$$= \frac{1}{4} [2 \log s - \log (s^2 + 4)]_s^{\infty}$$

$$= \frac{1}{4} \left[\log \left(\frac{s^2}{s^2 + 4} \right) \right]_s^{\infty}$$

$$= \frac{1}{4} \left[\log \left(\frac{1}{1 + \frac{4}{s^2}} \right) \right]_s^{\infty}$$

$$= \frac{1}{4} \left[\log 1 - \log \left(\frac{1}{1 + \frac{4}{s^2}} \right) \right]$$

$$= -\frac{1}{4} \log \left(\frac{s^2}{s^2 + 4} \right)$$

$$= \frac{1}{4} \log \left(\frac{s^2 + 4}{s^2} \right)$$

$$\text{From (i)} \Rightarrow I = \left[\frac{1}{4} \log \left(\frac{s^2 + 4}{s^2} \right) \right]_{s=1}$$

$$= \frac{1}{4} \log \left(\frac{5}{1} \right)$$

$$= \frac{1}{4} \log 5.$$

Q. 2. (b) State second shifting theorem for Laplace transform and hence find the Laplace transform of the following function

$$F(t) = \begin{cases} e^{t-a}, & t > a \\ 0, & t < a \end{cases}$$

Ans. Second Shifting Theorem

IF $L\{F(t)\} = f(s)$

$$\text{and } G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

$$\text{then } L\{G(t)\} = e^{-as} f(s) = e^{-as} L\{F(t)\}$$

$$\text{Given } F(t) = \begin{cases} e^{t-a}, & t > a \\ 0, & t < a \end{cases}$$

$$L\{F(t)\} = e^{-as} L\{e^t\} = e^{-as} \frac{1}{s-a}$$

Q. 2. (c) Using convolution theorem, find the inverse Laplace transform of the following $\frac{s}{(s^2 + a^2)^3}$.

$$\text{Ans. Let } f(s) = \frac{s}{(s^2 + a^2)^2},$$

$$g(s) = \frac{1}{(s^2 + a^2)}$$

$$L^{-1}\left\{\frac{1}{s^2 + a^2}\right\} = \frac{1}{a} \sin at$$

$$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2 + a^2}\right)\right\} = -\frac{t}{a} \sin at$$

$$L^{-1}\left\{\frac{-2s}{(s^2 + a^2)^2}\right\} = -\frac{t}{a} \sin at$$

$$L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{t}{2a} \sin at$$

$$L^{-1}\{f(s)\} = L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\}$$

$$= \frac{t}{2a} \sin at = F(t)$$

$$L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{s^2 + a^2}\right\}$$

$$= \frac{1}{a} \sin at = G(t)$$

By Convolution theorem, we have

$$L^{-1}\left\{\frac{s}{(s^2 + a^2)^3}\right\} = \int_0^t F(u) G(t-u) du$$

$$= \int_0^t \frac{u}{2a} \sin au \frac{1}{a} \sin (at - au) du$$

$$= \frac{1}{2a^2} \int_0^t \frac{u}{2} \{2 \sin au \sin (at - au)\} du$$

$$= \frac{1}{4a^2} \int_0^t u \{\cos (2au - at) - \cos at\} du$$

$$= \frac{1}{4a^2} \left[\int_0^t u \cos (2au - at) du \right.$$

$$\left. - \int_0^t u \cos at du \right]$$

$$= \frac{1}{4a^2} \left[u \left(\frac{\sin (2au - at)}{2a} \right) \right.$$

$$\left. - 1 \left\{ \frac{\cos (2au - at)}{4a^2} \right\} - \frac{u^2}{2} \cos at \right]_0^t$$

$$= \frac{1}{4a^2} \left[\frac{t}{2a} \sin at + \frac{\cos at}{4a^2} \right.$$

$$\left. - \frac{t^2}{2} \cos at - \frac{\cos at}{4a^2} \right]$$

$$= \frac{1}{4a^2} \left[\frac{t}{2a} \sin at - \frac{t^2}{2} \cos at \right]$$

$$= \frac{t}{8a^3} (\sin at - at \cos at)$$

Q. 2. (d) Solve the following simultaneous differential equations by Laplace transform

$$\frac{dx}{dt} + 4 \frac{dy}{dt} - y = 0$$

$$\frac{dx}{dt} + 2y = e^{-t}$$

with condition $x(0) = y(0) = 0$.

$$\text{Ans. } \Rightarrow x' + 4y' - y = 0$$

$$x' + 2y = e^{-t}$$

$$x(0) = y(0) = 0$$

Taking Laplace transform

$$\Rightarrow L(x') + 4L(y') - L(y) = 0$$

$$\Rightarrow sL(x) - x(0) + 4[sL(y) - y(0)] - L(y) = 0$$

$$sL(x) - x(0) + 2L(y) = \frac{1}{s-1}$$

Given $x(0) = y(0) = 1$ & Let $L(x) = \bar{x}$, $L(y) = \bar{y}$

$$\Rightarrow s\bar{x} + 4s\bar{y} - \bar{y} = 0$$

$$s\bar{x} + 2\bar{y} = \frac{1}{s-1}$$

$$\Rightarrow s\bar{x} + (4s-1)\bar{y} = 0 \quad \dots(1)$$

$$s\bar{x} + 2\bar{y} = \frac{1}{s-1} \quad \dots(2)$$

Eliminating \bar{y} between (1) & (2) we get

$$(2s - s(4s-1))\bar{x} = -\left(\frac{4s-1}{s-1}\right)$$

$$(2s - 4s^2 + s)\bar{x} = -\left(\frac{4s-1}{s-1}\right)$$

$$-(4s^2 - 3s)\bar{x} = -\left(\frac{4s-1}{s-1}\right)$$

$$\bar{x} = \frac{(4s-1)}{s(s-1)(4s-3)}$$

$$\frac{4s-1}{s(s-1)(4s-3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{4s-3}$$

$$s=0, A = -\frac{1}{3}$$

$$s=1, B = 3$$

$$s = \frac{3}{4}, C = -\frac{32}{3}$$

$$\bar{x} = -\frac{1}{3s} + \frac{3}{s-1} - \frac{32}{3(4s-3)}$$

$$L(x) = -\frac{1}{3s} + \frac{3}{s-1} - \frac{32}{12\left(s - \frac{3}{4}\right)}$$

Taking inverse Laplace transform, we get

$$x = -\frac{1}{3} + 3e^t - \frac{8}{3}e^{3t/4} \quad \dots(3)$$

$$[4s-1-2]\bar{y} = -\frac{1}{s-1}$$

$$\bar{y} = -\frac{1}{(s-1)(4s-3)}$$

$$\frac{1}{(s-1)(4s-3)} = \frac{A}{s-1} + \frac{B}{4s-3}$$

$$s=1, A=1$$

$$s = \frac{3}{4}, B = -4$$

$$L(y) = \bar{y} = -\frac{1}{s-1} + \frac{4}{4s-3}$$

Taking inverse Laplace transform, we get

$$y = -e^t + \frac{4}{3}e^{3t/4}$$

$$y = -e^t + e^{3t/4} \quad \dots(4)$$

Hence

$$x = -\frac{1}{3} + 3e^t - \frac{8}{3}e^{3t/4}$$

$$y = -e^t + e^{3t/4}$$

Q. 2. (e) Solve the following differential equation using Laplace transform

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x} \sin x$$

where $y(0) = 0$, $\left(\frac{dy}{dx}\right)_{x=0} = 1$.

Ans.

$$y'' + 2y' + 5y = e^{-x} \sin x$$

Taking Laplace transform; we get

$$L(y'') + 2L(y') + 5L(y) = L(e^{-x} \sin x)$$

$$(s^2L(y) - 5y(0) - y'(0)) + 2(sL(y) - y(0))$$

$$+ 5L(y) = \frac{1}{(s+1)^2 + 1}$$

$$(s^2 + 2s + 5)L(y) - 1 = \frac{1}{s^2 + 2s + 2}$$

$$L(y) = \frac{1}{s^2 + 2s + 5}$$

$$\begin{aligned}
&= \frac{1}{s^2 + 2s + 5} \\
&\quad + \frac{1}{3} \left[\frac{1}{s^2 + 2s + 2} - \frac{1}{s^2 + 2s + 5} \right] \\
L(y) &= \frac{1}{3} \frac{1}{s^2 + 2s + 2} + \frac{2}{3} \frac{1}{s^2 + 2s + 5} \\
\text{Taking inverse laplace transform} \\
y &= \frac{1}{3} L^{-1} \left(\frac{1}{s^2 + 2s + 2} \right) \\
&\quad + \frac{2}{3} L^{-1} \left(\frac{1}{s^2 + 2s + 5} \right) \\
&= \frac{1}{3} L^{-1} \left[\frac{1}{(s+1)^2 + 1} \right] \\
&\quad + \frac{2}{3} L^{-1} \left[\frac{1}{(s+1)^2 + 4} \right] \\
&= \frac{1}{3} e^{-x} L^{-1} \left(\frac{1}{s^2 + 1} \right) + \frac{2}{3} e^{-x} L^{-1} \left(\frac{1}{s^2 + 4} \right) \\
&= \frac{e^{-x}}{3} \sin x + \frac{2}{3} e^{-x} \cdot \frac{1}{2} \sin 2x \\
&= \frac{e^{-x}}{3} (\sin x + \sin 2x)
\end{aligned}$$

Q. 2. (f) Using unit step function, find the Laplace transform of:

(i) $(t-1)^2 \cdot u(t-1)$

(ii) $\sin t \cdot u(t-\pi)$

Ans. (i) $L\{(t-1)^2 \cdot u(t-1)\} = e^{-s} L^{-1}\{t^2\}$

$$\begin{aligned}
&= e^{-s} \cdot \frac{2}{s^3} \\
&= \frac{2e^{-s}}{s^3}
\end{aligned}$$

(ii) $L\{\sin t \cdot u(t-\pi)\}$

$$\begin{aligned}
&= L\{\sin[(t-\pi) + \pi] u(t-\pi)\} \\
&= -L\{\sin(t-\pi) u(t-\pi)\} \\
&= -e^{-\pi s} L\{\sin t\} = \frac{-e^{-\pi s}}{s^2 + 1}
\end{aligned}$$

Q. 3. Attempt any two parts of the following: $10 \times 2 = 20$

(a) Solve the following differential equation in series:

$$2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + (x-5)y = 0$$

Ans. (a) $2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + (x-5)y = 0$... (1)

Here, $x=0$ is a regular singular point.

Let $y = \sum_{k=0}^{\infty} a_k x^{m+k}$... (2)

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2}$$

Substituting these values in the equation (1), we get

$$\begin{aligned}
&2x^2 \sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2} \\
&- x \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1} \\
&+ (x-5) \sum_{k=0}^{\infty} a_k x^{m+k} = 0
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{\infty} a_k [2(m+k)(m+k-1) \\
&\quad - (m+k) - 5] x^{m+k} \\
&+ \sum_{k=0}^{\infty} a_k x^{m+k+1} = 0.
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{\infty} a_k (2m+2k-5)(m+k-1) x^{m+k} \\
&+ \sum_{k=0}^{\infty} a_k x^{m+k+1} = 0 \quad \dots (3)
\end{aligned}$$

Now, equate to zero the coefficient of lowest power of x , we get

$$a_0(2m-5)(m-1) = 0$$

$$\boxed{m = \frac{5}{2}, 1} \quad \therefore a_0 \neq 0$$

The roots of indicial equation are distinct and do not differ by an integer.

Now, equate to zero the coefficient of next power of x , we get

$$a_1(2m+2-5)(m+1-1) + a_0 = 0$$

$$a_1(2m-3)(m) + a_0 = 0$$

$$a_1 = -\frac{a_0}{m(2m-3)}$$

Now, equate to zero the coefficient of x^{m+k+1} , we get

$$a_{k+1}(2m+2k+2-5)(m+k+1-1) + a_k = 0$$

$$\boxed{a_{k+1} = -\frac{a_k}{(m+k)(2m+2k-3)}} \quad \dots(4)$$

$$k=1, a_2 = -\frac{a_1}{(m+1)(2m-1)}$$

$$= -\frac{a_0}{m(m+1)(2m-1)(2m-3)}$$

$$k=2, a_3 = -\frac{a_2}{(m+2)(2m+1)}$$

$$= -\frac{a_0}{m(m+1)(m+2)(2m-1)(2m-3)(2m+1)}$$

and so on.

Hence, from (2),

$$y = x^m \left[a_0 - \frac{a_0 x}{m(2m-3)} + \frac{a_0 x^2}{m(m+1)(2m-1)(2m-3)} \dots \right]$$

$$y = a_0 x^m \left[1 - \frac{x}{m(2m-3)} + \frac{x^2}{m(m+1)(2m-1)(2m-3)} \dots \right]$$

...(5)

When $m=1$, from (5) \Rightarrow

$$y_1 = a_0 x \left[1 - \frac{x}{(-1)} + \frac{x^2}{2(1)(-1)} \dots \right]$$

$$y_1 = a_0 x \left[1 + x - \frac{x^2}{2} \dots \right] \quad \dots(6)$$

when $m = \frac{5}{2}$, from (5) \Rightarrow

$$y_2 = a_0 x^{5/2} \left[1 - \frac{x}{\frac{5}{2}(5-3)} + \frac{x^2}{\frac{5}{2} \cdot \frac{7}{2}(5-1)(5-3)} \dots \right]$$

$$= a_0 x^{5/2} \left[1 - \frac{x}{5} + \frac{x^2}{70} \dots \right]$$

Hence the complete solution is given

$$y = c_1 y_1 + c_2 y_2 = c_1 a_0 x \left(1 + x - \frac{x^2}{2} \dots \right) + c_2 a_0 x^{5/2} \left(1 - \frac{x}{5} + \frac{x^2}{70} \dots \right)$$

$$y = Ax \left(1 + x - \frac{x^2}{2} \dots \right) + B x^{5/2} \left(1 - \frac{x}{5} + \frac{x^2}{70} \dots \right)$$

Q. 3. (b) Show that

$$(1) \int_{-1}^{+1} [P_n(x)]^2 dx = \frac{2}{2n+1}$$

$$(2) P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2}$$

where $P_n(x)$ is the Legendre polynomial of degree n .

Ans. (i)

We know $(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$

Squaring both sides, we get

$$(1 - 2xz + z^2)^{-1} = \sum_{n=0}^{\infty} [z^n P_n(x)]^2 + 2 \sum_{m=0}^{\infty} \sum_{\substack{n=0 \\ (m \neq n)}}^{\infty} z^{m+n} P_m(x) P_n(x)$$

Integrating w.r.t x between the limit -1 to 1 , we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \int_{-1}^1 z^{2n} [P_n(x)]^2 dx \\ & + 2 \sum_{m=0}^{\infty} \sum_{\substack{n=0 \\ m \neq n}}^{\infty} \int_{-1}^1 z^{m+n} P_m(x) P_n(x) dx \\ & = \int_{-1}^1 \frac{dx}{1 - 2xz + z^2} \\ & \sum_{n=0}^{\infty} \int_{-1}^1 z^{2n} [P_n(x)]^2 dx \\ & = -\frac{1}{2z} [\log(1 - 2xz + z^2)]_{-1}^1 \\ & \sum_{n=0}^{\infty} z^{2n} \int_{-1}^1 [P_n(x)]^2 dx \\ & = -\frac{1}{2z} [\log(1 - z)^2 - \log(1 + z)^2] \\ & = \frac{1}{z} [\log(1 + z) - \log(1 - z)] \\ & = \frac{1}{z} \log \left(\frac{1+z}{1-z} \right) \\ & = \frac{2}{z} \left(z + \frac{z^3}{3} + \frac{z^5}{5} + \dots \right) \\ & \sum_{n=0}^{\infty} z^{2n} \int_{-1}^1 [P_n(x)]^2 dx \\ & = \frac{2}{z} \left(z + \frac{z^3}{3} + \frac{z^5}{5} + \dots + \frac{z^{2n+1}}{2n+1} + \dots \right) \end{aligned}$$

Equating the coefficient of z^{2n} on the two sides, we get

$$\int_{-1}^1 [P_n(x)]^2 dx = \frac{2}{2n+1}$$

$$(2) P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2}$$

$$\sum_{n=0}^{\infty} z^n P_n(x) = (1 - 2xz + z^2)^{-1/2}$$

Putting $x = 0$, we get

$$\begin{aligned} \sum_{n=0}^{\infty} z^n P_n(0) &= (1 + z^2)^{-1/2} \\ &= 1 - \frac{1}{2} z^2 + \frac{13}{24} z^4 + \dots \\ &\dots + (-1)^r \frac{135 \dots (2r-1)}{246 \dots (2r)} z^{2r} + \dots \end{aligned}$$

Equating the coefficient of z^{2n} on both sides, we get

$$\begin{aligned} P_{2n}(0) &= (-1)^n \frac{135 \dots (2n-1)}{246 \dots (2n)} \\ &= (-1)^n \frac{1 \cdot 2 \cdot 3 \cdot 4 \dots (2n-1)(2n)}{[2 \cdot 4 \cdot 6 \dots (2n)]^2} \end{aligned}$$

$$\begin{aligned} P_{2n}(0) &= (-1)^n \frac{(2n)!}{[2^n \cdot 1 \cdot 2 \cdot 3 \dots n]^2} \\ &= (-1)^n \frac{(2n)!}{2^{2n} (n!)^2} \end{aligned}$$

Q. 3. (c) Show that

$$(1) J_{\frac{1}{2}}(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x)$$

$$(2) J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \dots) = 1$$

where $J_n(x)$ is the Bessel's polynomial of degree n and dash denote the differentiation.

$$\text{Ans. } J_{\frac{1}{2}}(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x)$$

By Recurrence relation

$$x J_n' = -n J_n + x J_{n-1} \dots (1)$$

putting $n = 2$

$$xJ_2^1 = -2J_2 + xJ_1$$

$$\Rightarrow J_2^1 = -\frac{2}{x}J_2 + J_1 \quad \dots(2)$$

By Recurrence relation

$$xJ_n^1 = nJ_n - xJ_{n+1} \quad \dots(3)$$

From (1) and (3), we have

$$-nJ_n + xJ_{n-1} = nJ_n - xJ_{n+1}$$

putting $n = 1$

$$-J_1 + xJ_0 = J_1 - xJ_2$$

$$J_2 = \frac{2}{x}J_1 - J_0 \quad \dots(4)$$

\(\therefore\) From (2) and (4)

$$J_2 = -\frac{2}{x}\left(\frac{2}{x}J_1 - J_0\right) + J_1$$

$$J_2^1 = \left(1 - \frac{4}{x^2}\right)J_1 + \frac{2}{x}J_0$$

$$(2) J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \dots) = 1$$

The Jacobi series are

$$J_0(x) + 2J_2(x) \cos 2\theta + 2J_4(x) \cos 4\theta + \dots = \cos(x \sin \theta) \quad \dots(1)$$

$$2J_1(x) \sin \theta + 2J_3(x) \sin 3\theta + \dots = \sin(x \sin \theta) \quad \dots(2)$$

Squaring (1) and (2) and integrating

w.r.t. to θ between the limit 0 and π ,

$$\int_0^\pi \cos^2 n\theta \, d\theta = \int_0^\pi \sin^2 \theta \, d\theta = \frac{\pi}{2}$$

$$\text{and } \int_0^\pi \cos m\theta \cos n\theta \, d\theta$$

$$= \int_0^\pi \sin m\theta \sin n\theta \, d\theta = 0, \quad m \neq n$$

$$[J_0(x)]^2 + 2[J_2(x)]^2 \pi + 2[J_4(x)]^2 \pi + \dots = \int_0^\pi \cos^2(x \sin \theta) \, d\theta$$

$$2[J_1(x)]^2 \pi + 2[J_3(x)]^2 \pi + \dots = \int_0^\pi \sin^2(x \sin \theta) \, d\theta$$

on adding, we get

$$\pi[J_0^2 + 2J_1^2 + J_2^2 + J_3^2 + J_4^2 + \dots]$$

$$= \int_0^\pi [\cos^2(x \sin \theta) + \sin^2(x \sin \theta)] \, d\theta$$

$$= \int_0^\pi d\theta = (\theta)_0^\pi$$

$$\pi[J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + J_4^2 + \dots)] = \pi$$

$$J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + J_4^2 + \dots) = 1$$

Q. 4. Attempt any two parts of the following: $10 \times 2 = 20$

(a) Obtain a Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$ and hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

$$\text{Ans. } f(x) = x^2$$

$$\text{Let } f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$+ \sum_{n=1}^{\infty} b_n \sin nx \quad \dots(1)$$

$$\text{Where } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \, dx = \frac{1}{\pi} \left(\frac{x^3}{3} \right)_0^{2\pi}$$

$$= \frac{1}{\pi} \left(\frac{8\pi^3}{3} - 0 \right) = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx \, dx$$

$$= \frac{1}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - (2x) \left(-\frac{\cos nx}{n^2} \right) \right.$$

$$\left. + (2) \left(-\frac{\sin nx}{n^3} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{4\pi}{n^2} \cos 2n\pi \right]$$

$$= \frac{4}{n^2} (-1)^{2n} = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx \, dx$$

$$= \frac{1}{\pi} \left[x^2 \left(-\frac{\cos nx}{n} \right) - (2x) \left(-\frac{\sin nx}{n^2} \right) \right. \\ \left. + (2) \left(\frac{\cos nx}{n^3} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\left(-\frac{4\pi^2}{n} \cos 2n\pi + \frac{2}{n^3} \cos 2n\pi \right) \right. \\ \left. - \left(\frac{2}{n^3} \right) \right]$$

$$= \frac{1}{\pi} \left[-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3} \right]$$

$$= -\frac{4\pi}{n}$$

from (1) \Rightarrow

$$x^2 = \frac{1}{2} \left(\frac{8\pi^2}{3} \right) + 4 \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$-4\pi \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

$$x^2 = \frac{4\pi^2}{3}$$

$$+ 4 \left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots \right)$$

$$-4\pi \left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right)$$

put $x = \pi$

$$\pi^2 = \frac{4\pi^2}{3}$$

$$+ 4 \left(\frac{\cos \pi}{1^2} + \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} + \frac{\cos 4\pi}{4^2} + \dots \right)$$

$$\pi^2 = \frac{4\pi^2}{3}$$

$$= 4 \left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \dots \right)$$

$$-\frac{\pi^2}{3} = -4 \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Q. 4. (b) Examine whether the function $f(x) = x \sin x$ is even or odd. Hence expand it in the form of Fourier series in the interval $(-\pi, \pi)$.

Ans. $f(x) = x \sin x$

$$f(-x) = (-x) \sin(-x) = x \sin x = f(x)$$

Since $f(x)$ is an even function of x ,

$$b_n = 0$$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \dots(1)$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \sin x \, dx$$

$$= \frac{2}{\pi} [x(-\cos x) - 1(-\sin x)]_0^{\pi}$$

$$= \frac{2}{\pi} (-\pi \cos \pi) = 2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x (2 \cos nx \sin x) \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x [\sin(n+1)x - \sin(n-1)x] \, dx$$

$$= \frac{1}{\pi} \left[x \left\{ -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right\} \right. \\ \left. - 1 \left\{ -\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\pi \left\{ -\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} \right\} \right]$$

$$a_n = \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}, n \neq 1$$

when n is odd, $n \neq 1$

$$\therefore a_n = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1}$$

when n is even, $n \neq 1$

$$\therefore a_n = \frac{-1}{n-1} + \frac{1}{n+1} = -\frac{2}{n^2-1}$$

when $n = 1$,

$$a_1 = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi} \left[x \left(-\frac{\cos 2x}{2} \right) - 1 \left(-\frac{\sin 2x}{4} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\pi \frac{\cos 2\pi}{2} \right] = -\frac{1}{2}$$

$$\therefore x \sin x = 1 - \frac{1}{2} \cos x$$

$$- 2 \left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \dots \right)$$

...(2)

$$x \sin x = 1 - \frac{\cos x}{2}$$

$$- 2 \left(\frac{\cos 2x}{3} - \frac{\cos 3x}{8} + \frac{\cos 4x}{15} - \dots \right)$$

Q. 4. (c) Solve the following partial differential equations :

(1) $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$

(2) $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} = \sin x \cdot \cos 2y$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

Ans.

$$x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$$

Here the auxiliary equation are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}$$

using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz$$

Each fraction = $\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz$

$$\therefore \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0$$

On integration gives

$$\log x + \log y + \log z = \log a$$

$$xyz = a \quad \dots(1)$$

Again using $x, y, -1$ as multipliers we

get

$$\text{each fraction} = \frac{xdx + ydy - dz}{0}$$

$$\therefore xdx + ydy - dz = 0$$

On integration gives

$$\frac{x^2}{2} + \frac{y^2}{2} - z = b$$

$$\frac{x^2 + y^2 - 2z}{2} = b \quad \dots(2)$$

From eqn. (1) & (2), the general solution is

$$\phi \left(xyz, \frac{x^2 + y^2 - 2z}{2} \right) = 0 \text{ Ans}$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{2\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$$

$$(D^2 - 2DD')z = \sin x \cos 2y$$

A.E. is $m^2 - 2m = 0$

$$m(m-2) = 0$$

$$m = 0, 2$$

C.F. = $f_1(y) + f_2(y + 2x)$

$$\text{P.I.} = \frac{1}{D^2 - 2DD'} (\sin x \cos 2y)$$

$$= \frac{1}{1} \cdot \frac{1}{1} (2 \sin x \cos 2y)$$

$$= \frac{1}{2} \frac{1}{D^2 - 2DD'} [\sin(x + 2y) + \sin(x - 2y)]$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - 2DD'} \sin(x + 2y) + \frac{1}{D^2 - 2DD'} \sin(x - 2y) \right]$$

$$= \frac{1}{2} \left[\frac{1}{-1 - 2(-2)} \sin(x + 2y) + \frac{1}{-1 - 2(2)} \sin(x - 2y) \right]$$

$$= \frac{1}{2} \left[\frac{1}{3} \sin(x + 2y) - \frac{1}{5} \sin(x - 2y) \right]$$

$$= \frac{1}{6} \sin(x + 2y) - \frac{1}{10} \sin(x - 2y)$$

The complete solution is

$z = C.F + P.I$

$$z = f_1(y) + f_2(y + 2x) + \frac{1}{6} \sin(x + 2y) - \frac{1}{10} \sin(x - 2y)$$

Q. 5. Attempt any two parts of the following : $10 \times 2 = 20$

(a) Solve the following equation by the method of separation of variables :

$$\frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y} \text{ where } u(0, y) = 8e^{-3y}$$

$$\text{Ans. } \frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y} \quad \dots(1)$$

$$u(0, y) = 8e^{-3y}$$

$$\text{Let } u = XY \quad \dots(2)$$

Where X is a function of x only and Y is a function of y only.

$$X'Y = 4XY'$$

$$\frac{X'}{X} = 4 \frac{Y'}{Y} = -p^2 \text{ (say)}$$

$$(i) \frac{X'}{X} = -p^2 \Rightarrow \frac{dX}{dx} = -p^2 X$$

$$\frac{dX}{X} = -p^2 dx$$

Integrating, $\log X = -p^2 x + \log c_1$

$$X = c_1 e^{-p^2 x} \quad \dots(3)$$

$$(ii) \frac{4Y'}{Y} = -p^2$$

$$\frac{dY}{dy} = -\frac{p^2}{4} Y$$

$$\frac{dY}{Y} = -\frac{p^2}{4} dy$$

Integrating, $\log Y = -\frac{p^2}{4} y + \log c_2$

$$Y = c_2 e^{-p^2 y/4} \quad \dots(4)$$

From (2), (3) and (4), we get

$$u(x, y) = c_1 c_2 e^{-p^2 x} \cdot e^{-p^2 y/4}$$

$$u(x, y) = c_1 c_2 e^{-p^2(x + y/4)} \quad \dots(5)$$

Given $u(0, y) = 8e^{-3y}$

$$8e^{-3y} = c_1 c_2 e^{-p^2 y/4}$$

$$\Rightarrow c_1 c_2 = 8, \frac{p^2}{4} = 3$$

$$p^2 = 12$$

from (5),

$$u(x, y) = 8e^{-12(x + y/4)} = 8e^{-(12x + 3y)}$$

Q. 5. (h) Solve the following Laplace equation :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in a rectangle with $u(0, y) = 0$, $u(a, y) = 0$; $u(x, b) = 0$ and $u(x, 0) = f(x)$ along x -axis.

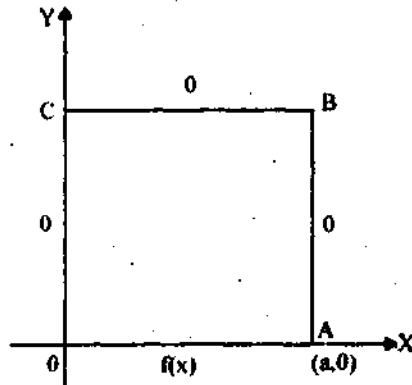
$$\text{Ans. } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(1)$$

B.C are $u(0, y) = 0$

$$u(a, y) = 0$$

$$u(x, b) = 0$$

$$u(x, 0) = f(x)$$



Let $u = X Y$... (2)

where X is a function of x only and Y is a function of y only then, from (1) \Rightarrow

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -p^2$$

$$\frac{X''}{X} = -p^2$$

$$X'' + p^2 X = 0$$

$$\Rightarrow \frac{d^2 X}{dx^2} + p^2 X = 0$$

A.E. is $m^2 + p^2 = 0 \Rightarrow m = \pm pi$

$$X = c_1 \cos px + c_2 \sin px$$

$$-\frac{Y''}{Y} = -p^2$$

$$\Rightarrow \frac{Y''}{Y} = p^2$$

$$\frac{d^2 Y}{dy^2} - p^2 Y = 0$$

A.E. is $m^2 - p^2 = 0$

$$m = \pm p$$

$$Y = c_3 e^{py} + c_4 e^{-py}$$

So, $u(x, y) = XY$

$$= (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py})$$

Using B.C. $u(0, y) = 0$ in (3), we get

$$0 = (c_1 + 0)(c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow c_1 = 0$$

using B.C. $u(a, y) = 0$ in (3), we get

$$0 = (c_1 \cos pa + c_2 \sin pa)(c_3 e^{py} + c_4 e^{-py})$$

$$c_2 \sin pa = 0 \Rightarrow \sin pa = 0 = \sin n\pi$$

$$p = \frac{n\pi}{a}$$

From (3),

$$u(x, y) = c_2 \sin \frac{n\pi x}{a} \left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}} \right)$$

$$u(x, y) = \sin \frac{n\pi x}{a} \left(A e^{\frac{n\pi y}{a}} + B e^{-\frac{n\pi y}{a}} \right)$$

where $c_2 c_3 = A$ and $c_2 c_4 = B$.

$u(x, b) = 0$ in (4), we get

$$u(x, b) = \sin \frac{n\pi x}{a} \left(A e^{\frac{n\pi b}{a}} + B e^{-\frac{n\pi b}{a}} \right)$$

$$0 = \sin \frac{n\pi x}{a} \left(A e^{\frac{n\pi b}{a}} + B e^{-\frac{n\pi b}{a}} \right)$$

$$\Rightarrow A e^{\frac{n\pi b}{a}} + B e^{-\frac{n\pi b}{a}} = 0$$

$$A e^{\frac{n\pi b}{a}} = -B e^{-\frac{n\pi b}{a}} = -\frac{1}{2} B_n \text{ (say)}$$

(4) becomes,

$$u(x, y) = \sin \frac{n\pi x}{a} \left[-\frac{1}{2} B_n e^{\frac{-n\pi b}{a}} e^{\frac{n\pi y}{a}} + \frac{1}{2} B_n e^{\frac{n\pi b}{a}} e^{-\frac{n\pi y}{a}} \right]$$

$$= \frac{1}{2} B_n \sin \frac{n\pi x}{a} \left[e^{\frac{n\pi}{a}(b-y)} - e^{-\frac{n\pi}{a}(b-y)} \right]$$

$$= \frac{1}{2} B_n \sin \frac{n\pi x}{a} 2 \sinh \frac{n\pi}{a} (b-y)$$

$$= B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a} (b-y)$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a} (b-y) \quad \dots(5)$$

using B.C. $u(x, 0) = f(x)$, we get

from (5), $u(x, 0) = f(x)$

$$= \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

$$B_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

$$B_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a} dx \quad \dots(6)$$

Equation (5) gives the required solution of equation (1) where B_n can be determined by equation (6).

Q. 5. (c) Assuming the resistance of wire (R) and conductance to ground (C) are negligible, find the voltage $v(x, t)$ and current $i(x, t)$ in a transmission line of length l , t seconds after the ends are suddenly grounded. The initial conditions are $v(x, 0) = v_0 \sin\left(\frac{\pi x}{l}\right)$ and $i(x, 0) = i_0$.

Ans. Since R and C , are negligible, transmission line equations becomes

$$\frac{\partial v}{\partial x} = -L \frac{\partial i}{\partial t} \quad \dots(1)$$

$$\frac{\partial i}{\partial x} = -C \frac{\partial v}{\partial t} \quad \dots(2)$$

For elimination of i , differentiating (1) partially w.r. to x and (2) partially w.r. to t , we have

$$\frac{\partial^2 v}{\partial x^2} = -L \frac{\partial^2 i}{\partial x \partial t} \text{ and } \frac{\partial^2 i}{\partial t \partial x} = -C \frac{\partial^2 v}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2} \quad \dots(3)$$

The initial conditions are

$$i(x, 0) = i_0, v(x, 0) = v_0 \sin\left(\frac{\pi x}{l}\right) \quad \dots(4)$$

Since the ends are suddenly grounded, the boundary conditions are

$$v(0, t) = v(l, t) = 0 \quad \dots(5)$$

Also $i = i_0$ (constant) when $t = 0$

$$\frac{\partial v}{\partial x} = 0 \text{ which gives } \frac{\partial v}{\partial t} = 0 \text{ when } t = 0 \quad \dots(6)$$

(using (2))

Now let $V = XT$

be the a solution of (3), where X is a function of x only and T is a function of t only.

from (3), $X''T = LCXT''$

$$\frac{X''}{X} = LC \frac{T''}{T} = -p^2 \text{ (say)}$$

$$\frac{X''}{X} = -p^2 \Rightarrow \frac{d^2 X}{dx^2} + p^2 X = 0$$

$$\Rightarrow X = c_1 \cos px + c_2 \sin px$$

$$LC \frac{T''}{T} = -p^2 \Rightarrow \frac{d^2 T}{dt^2} + \frac{p^2}{LC} T = 0$$

$$\Rightarrow T = c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}}$$

$$\Rightarrow v = XT = (c_1 \cos px + c_2 \sin px)$$

$$\left(c_3 \cos \frac{pt}{\sqrt{LC}} + c_4 \sin \frac{pt}{\sqrt{LC}} \right) \quad \dots(7)$$

Applying B.C. (5), we have

$$c_1 = 0$$

and $p = \frac{n\pi}{l}$, n being an integer.

Equation (7) \Rightarrow

$$v = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi t}{\sqrt{LC}} + c_4 \sin \frac{n\pi t}{\sqrt{LC}} \right)$$

$$v = \sin \frac{n\pi x}{l} \left(A \cos \frac{n\pi t}{\sqrt{LC}} + B \sin \frac{n\pi t}{\sqrt{LC}} \right) \quad \dots(8)$$

where $c_2c_3 = A$ and $c_2c_4 = B$.

$$\frac{\partial v}{\partial t} = \sin \frac{n\pi x}{l}$$

$$\left(-\frac{Ah\pi}{\sqrt{LC}} \sin \frac{n\pi t}{\sqrt{LC}} + \frac{Bn\pi}{\sqrt{LC}} \cos \frac{n\pi t}{\sqrt{LC}} \right)$$

since $\frac{\partial v}{\partial t} = 0$ when $t = 0$, we get,

$$B = 0$$

$$\text{from (8), } v = A \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{\sqrt{LC}}$$

By superposition, we get

$$v = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{\sqrt{LC}}$$

Applying I.C. $v = v_0 \sin \frac{\pi x}{l}$ when $t = 0$

$$v_0 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow A_1 = v_0 \text{ and } A_2 = A_3 = \dots = 0$$

$$\text{Hence, } v = v_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{\sqrt{LC}}$$

Now, from (1),

$$-L \frac{\partial i}{\partial t} = \frac{\partial v}{\partial x}$$

$$\frac{\partial i}{\partial t} = -\frac{1}{L} \frac{v_0 \pi}{l} \cos \frac{\pi x}{l} \cos \frac{\pi t}{\sqrt{LC}}$$

Integrating w.r. to t , regarding x as constant.

$$i = -\frac{v_0 \pi}{Ll} \cos \frac{\pi x}{l} \cdot \frac{\sqrt{LC}}{\pi} \sin \frac{\pi t}{\sqrt{LC}} + f(x) \quad \dots(9)$$

Where $f(x)$ is an arbitrary constant function. Since $i = i_0$ when $t = 0$, we have

$$i_0 = 0 + f(x) \Rightarrow f(x) = i_0$$

from (9),

$$i = i_0 - v_0 \sqrt{\frac{c}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi t}{\sqrt{LC}}$$