B.Tech. EAS-203

Second Semester Examination 2009-10

Mathematics-II

Section-A

Note : Attempt all questions. Each question carries equal marks. (10 × 2 = 20)

Q. 1. (a) The order and degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^4 - 6x^2\left(\frac{dy}{dx}\right)^8 = 0$ are ...... and ....

Ans. Order = 3, Degree = 4

Q. 1. (b) Pick the correct statement from the following :

(i) Integrating factor to a differential equation is unique.

(ii) $y = ex$ is the general solution of the $\frac{ydx - xdy}{x^2} = 0$.

Ans. (ii) $y = ex$ is the general solution of the $\frac{ydx - xdy}{x^2} = 0$.

Q. 1. (c) Write the Rodrigue formula $P_n(x) = ...$

Ans. Rodrigue formula $P_n(x) = \frac{1}{2^n\ n!}\frac{d^n}{dx^n}(x^2 - 1)^n$

Q. 1. (d) If J_0 and J_1 are Bessel's functions, then $J_1(x)$ is given by :

(i) $J_0(x) - \frac{1}{x}J_1(x)$ **(ii) $J_0 + \frac{1}{x}J_1(x)$**

(iii) $-J_0(x)$ **(iv) None of these**

Ans. (iv) None of these

Q. 1. (e) $L^{-1}\left\{\frac{1}{S^n}\right\}$ exist only when the value of n is

(i) Negative integer **(ii) Positive integer**

(iii) Zero **(iv) None of these**

Ans. (ii) Positive integer

Q 1. (f) Pick the correct statement for final value theorem of Laplace transform :

(i) $\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ sF(s)$ **(ii) $\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$**

Ans. (ii) $\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$

Q. 1. (g) Fourier coefficient 'a_0' in Fourier series expansion of a function represents the :

(i) maximum value of the function **(ii) mean value of the function**

(iii) minimum value of the function **(iv) none of these**

Ans. (ii) Mean value of the function.

Q. 1. (h) If the Fourier series of $f(x)$ has only cosine terms then $f(x)$ must be :

(i) odd function **(ii) even function**

Q. 1. (i) The PDE $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ is known as :

(i) wave equation **(ii) heat equation**

(iii) Laplace equation **(iv) none of these**

Ans. (i) Wave equation

Q. 1. (j) The PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, is :

(i) parabolic **(ii) elliptic**

(iii) hyperbolic **(iv) circular**

Ans. (ii) Elliptic

Section-B

*Note : Attempt **any three** parts from this section. Each part carry equal marks :* $(3 \times 10 = 30)$

Q. 2. (a) A particle of mass m moves in a straight line under the action of force mn^2x which is always directed towards a fixed point O on the line. Determine the displacement $x(t)$ if the resistance to the motion is $2\lambda mnv$ given that initially $x = 0$, $\frac{dx}{dt} = x_0$ $(0 < \lambda < 1)$.

Ans. The differential equation describing this simple harmonic motion is

$$m\frac{d^2x}{dt^2} = -2\lambda\, mnv - mn^2x, \quad \Rightarrow \quad m\frac{d^2x}{dt^2} = -2\lambda\, mn\frac{dx}{dt} - mn^2x$$

$$\frac{d^2x}{dt^2} + 2\lambda n\frac{dx}{dt} + n^2x = 0 \qquad \ldots(1)$$

The auxiliary equation is $M^2 + 2\lambda nM + n^2 = 0$

$$M = \frac{-2\lambda n \pm \sqrt{4\lambda^2n^2 - 4n^2}}{2} = -\lambda n \pm in\sqrt{1-\lambda^2}$$

The general solution is $x = e^{-\lambda nt}[c_1 \cos n\omega t + c_2 \sin n\omega t]$ ...(2)

where $\omega = \sqrt{1-\lambda^2}$,

using I.C. $x = 0, t = 0 \Rightarrow c_1 = 0$

Differentiating (2), we get

$$\frac{dx}{dt} = e^{-\lambda nt}[-c_1 n\,\omega \sin n\omega t + c_2 n\omega \cos n\omega t] - \lambda n e^{-\lambda nt}[c_1 \cos n\omega t + c_2 \sin n\omega t]$$

Again using I.C. $\frac{dx}{dt} = x_0, t = 0$

$$x_0 = -c_1 \cdot 0 + c_2 n\omega - \lambda n[c_1 + c_2 \cdot 0] \qquad c_2 = \frac{x_0}{n\omega}$$

From (2), its solution is $x = e^{-\lambda nt}\left(\frac{x_0}{n\omega}\right)\sin n\omega t$

Q. 2. (b) Using Frobenius method, obtain a series solution in powers of x for differential equation : $2x(1-x)\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + 3y = 0$

Ans. $2x(1-x)\dfrac{d^2y}{dx^2}+(1-x)\dfrac{dy}{dx}+3y=0$ ...(1)

$\because x=0$ is a regular singular point

Let $$y=x^m\sum_{n=0}^{\infty}a_n x^n=\sum_{m=0}^{\infty}a_n x^{m+n}$$ ...(2)

$$\frac{dy}{dx}=\sum_{n=0}^{\infty}a_n(m+n)x^{m+n-1}$$

$$\frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}a_n(m+n)(m+n-1)x^{m+n-2}$$

Substituting these values in equation (1), we get

$$2x[1-x]\sum_{n=0}^{\infty}a_n(m+n)(m+n-1)x^{m+n-2}+(1-x)\sum_{n=0}^{\infty}a_n(m+n)x^{m+n-1}+3\sum_{n=0}^{\infty}a_nx^{m+n}=0$$

$$\sum_{n=0}^{\infty}a_n[2(m+n)(m+n-1)+(m+n)]x^{m+n-1}+\sum_{n=0}^{\infty}a_n[-2(m+n)(m+n-1)-(m+n)+3]x^{m+n}=0$$

$$\sum_{m=0}^{\infty}a_n(m+n)(2m+2n-1)x^{m+n-1}-\sum_{n=0}^{\infty}a_n(m+n+1)(2m+2n-3)x^{m+n}=0$$ ...(3)

Coefficient of $x^{m-1}=0$

$$a_0(m)(2m-1)=0\Rightarrow a_0\neq 0$$

$$m(2m-1)=0\quad\Rightarrow\quad \boxed{m=0,\frac{1}{2}}$$

It is the root of indicial equation.

Coefficient of $x^m=0$

$$a_1(m+1)(2m+1)-a_0(m+1)(2m-3)=0\quad\Rightarrow\quad\boxed{a_1=\left(\frac{2m-3}{2m+1}\right)a_0}$$

Coefficient of $x^{m+n}=0$

$$a_{n+1}(m+n+1)(2m+2n+1)-a_n(m+n+1)(2m+2n-3)=0$$

$$\boxed{a_{n+1}=\left(\frac{2m+2n-3}{2m+2n+1}\right)a_n}$$ ...(4)

$n=1,\quad a_2=\left(\dfrac{2m-1}{2m+3}\right)a_1=\dfrac{(2m-1)(2m-3)}{(2m+1)(2m+3)}a_0$

$n=2,\quad a_3=\left(\dfrac{2m+1}{2m+5}\right)a_2=\dfrac{(2m-1)(2m-3)}{(2m+3)(2m+5)}a_0$

Substituting these value of a_1, a_2, a_3 ..... in equation (2), we get

$$y=a_0x^m\left[1+\left(\frac{2m-3}{2m+1}\right)x+\frac{(2m-1)(2m-3)}{(2m+1)(2m+3)}x^2+\frac{(2m-1)(2m-3)}{\ldots}x^3+\ldots\right]$$ ...(5)

Now, $y_1 = (y)_{m=0}$

$$y = a_0 x^0 \left[1 - 3x + \frac{3}{3}x^2 + \frac{3}{15}x^3 \ldots\right] = a_0\left[1 - 3x + x^2 + \frac{x^3}{5} + \ldots\right] \quad \ldots(6)$$

$$y_2 = (y)_{m=1/2} = a_0 x^{1/2}\left[1 - \frac{2}{2}x\right] = a_0\, x^{1/2}(1 - x)$$

Hence, the complete solution is, $y = c_1 y_1 + c_2 y_2$

$$= A\left(1 - 3x + x^2 + \frac{x^3}{5} \ldots\right) + Bx^{1/2}(1 - x)$$

where $A = c_1 a_0$ and $B = c_2 a_0$.

Q. 2. (c) Using Laplace transform, solve the differential equation :

$$\frac{d^2y}{dx^2} + n^2 y = a \sin(nx + 2)$$

Given, $y(0) = 0$ and $y'(0) = 0$.

Ans. $\frac{d^2y}{dx^2} + n^2 y = a\sin(nx + 2)$, Given $y(0) = 0$, $y'(0) = 0$

$$y'' + n^2 y = a(\sin nx \cos 2 + \cos nx \sin 2)$$

Taking Laplace transform both sides, we get

$$L(y'') + n^2 L(y) = a \cos 2 L(\sin nx) + a \sin 2 L(\cos nx)$$

$$s^2 L(y) - y(0) - y'(0) + n^2 L(y) = a\cos 2\left(\frac{n}{s^2 + n^2}\right) + a \sin 2\left(\frac{s}{s^2 + n^2}\right)$$

$$(s^2 + n^2)\, L(y) = a \cos 2\left(\frac{n}{s^2 + n^2}\right) + a\sin 2\left(\frac{s}{s^2 + n^2}\right)$$

$$L(y) = a \cos 2\left[\frac{n}{(s^2 + n^2)^2}\right] + a \sin 2\left[\frac{s}{(s^2 + n^2)^2}\right]$$

Taking inverse Laplace transform both sides,

$$y = a \cos 2 L^{-1}\left\{\frac{n}{(s^2 + n^2)^2}\right\} + a \sin 2 L^{-1}\left\{\frac{s}{(s^2 + n^2)^2}\right\} \quad \ldots(1)$$

$$L^{-1}\left\{\frac{s}{(s^2 + n^2)^2}\right\} = \frac{x}{2n}\sin nx$$

$$L^{-1}\left\{\frac{n}{(s^2 + n^2)^2}\right\} = \frac{1}{2n^2}(\sin nx - n\,x \cos nx)$$

$$y = a \cos 2\left[\frac{1}{2n^2}(\sin nx - nx \cos nx)\right] + a \sin 2\left(\frac{x}{2n}\sin nx\right)$$

Q. 2. (d) Find the Fourier series to represent the function $f(x)$ given by :

$$f(x) = \begin{cases} -K & \text{for} \quad -\pi < x < 0 \\ K & \text{for} \quad 0 < x < \pi \end{cases}$$

Hence show that : $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$.

Ans. $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\,dx + \int_{0}^{\pi}(K)\,dx\right]$

$$= \frac{1}{\pi}[-K(x)_{-\pi}^{0} + K(x)_{0}^{\pi}] = \frac{1}{\pi}[-K(-\pi - 0) + K(\pi - 0)] = \frac{1}{\pi}[K\pi + K\pi] = 2K$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \cdot dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\cos nx\,dx + \int_{0}^{\pi}K\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-K\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + K\left(\frac{\sin nx}{n}\right)_{0}^{\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\sin nx\,dx + \int_{0}^{\pi}K\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(-K)\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + K\left(\frac{-\cos nx}{n}\right)_{0}^{\pi}\right] = \frac{1}{\pi}\left[+\frac{K}{n}(1 - \cos n\pi) - \frac{K}{n}(\cos n\pi - 1)\right]$$

$$= \frac{K}{n\pi}[2(1 - \cos n\pi)] = \frac{2K}{n\pi}[1 - (-1)^n]$$

The Fourier series to the function $f(x)$

$$f(x) = \frac{2K}{2} - \frac{2K}{\pi}\sum_{n=1}^{\infty}\frac{[1 - (-1)^n]}{n}\sin nx$$

$$f(x) = K - \frac{2K}{\pi}\left[\frac{2\sin x}{1} + \frac{2\sin 3x}{3} + \frac{2\sin 5x}{5}\ldots\right]$$

$$f(x) = K - \frac{4K}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right]$$

put $x = \frac{\pi}{2}$

$$K = K\left[1 - \frac{4}{\pi}\left(\frac{\sin\frac{\pi}{2}}{1} + \frac{\sin\frac{3\pi}{2}}{3} + \frac{\sin\frac{5\pi}{2}}{5} + \ldots\right)\right]$$

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Q. 2. (e) Solve the Laplace's equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

in a rectangle in the xy-plane, $0 \le x \le a$ and $0 \le y \le b$ satisfying the following boundary

Ans. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ...(1)

Let $u = X(x)\,Y(y)$ ...(2)

where X is function of x only and Y is function of y only.

$(1) \Rightarrow \quad X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant}\,(+p^2)$$

$$\frac{X''}{X} = +p^2 \Rightarrow \frac{d^2X}{dx^2} - p^2 X = 0$$

Auxiliary equation is $m^2 - p^2 = 0 \Rightarrow m = \pm p$

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$-\frac{Y''}{Y} = p^2 \Rightarrow \frac{d^2Y}{dy^2} + p^2 Y = 0$$

Auxiliary equation is $m^2 + p^2 = 0 \Rightarrow m = \pm pi$

$$Y = c_3 \cos py + c_4 \sin py$$

$\therefore \quad u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$ ...(3)

Using B.C., $u(x, 0) = 0 \Rightarrow (c_1 e^{px} + c_2 e^{-px})\, c_3 = 0 \Rightarrow c_3 = 0$ ...(4)

Using B.C., $u(x, b) = 0 \Rightarrow (c_1 e^{px} + c_2 e^{-px})(c_3 \cos pb + c_4 \sin pb) = 0 \Rightarrow c_4 \sin pb = 0$

$\sin pb = 0 = \sin n\pi \Rightarrow \boxed{p = \frac{n\pi}{b}}$ ...(5)

Using B.C., $u(0, y) = 0 \Rightarrow (c_1 + c_2)(c_3 \cos py + c_4 \sin py) = 0 \Rightarrow \boxed{c_2 = -c_1}$ ...(6)

From equation (3), $u(x, y) = c_1 c_4 \left(e^{\frac{n\pi x}{b}} - e^{\frac{-n\pi x}{b}} \right) \sin \frac{n\pi y}{b}$

$$u(x, y) = 2c_1 c_4 \sin h\left(\frac{n\pi x}{b}\right) \sin \frac{n\pi y}{b}$$

The general solution is, $u(x, y) = \sum_{n=1}^{\infty} b_n \sin h\left(\frac{n\pi x}{b}\right) \sin \frac{n\pi y}{b}$ ...(7)

Using $u(a, y) = f(y)$

$$f(y) = \sum_{n=1} b_n \sin h\left(\frac{n\pi a}{b}\right) \sin \frac{n\pi y}{b},$$

$$\left(\sinh \frac{n\pi a}{b}\right) b_n = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy$$

$b_n = \frac{2}{b} \operatorname{cosech}\left(\frac{n\pi a}{b}\right) \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy$ ...(8)

Hence the required solution is given by (7) and b_n can be determined by (8).

Section-C

Note : Attempt any two parts from all questions of this section. Each part carry equal marks :

Q. 3. (a) Solve the following differential equation : $(5 \times 2 \times 5 = 50)$

Ans. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = \sin 3x + \cos 2x$

The auxiliary equation is $m^2 + 5m - 6 = 0$

$\Rightarrow (m+6)(m-1) = 0 \Rightarrow m = -6, 1$

$$\text{C.F.} = c_1e^{-6x} + c_2e^{x}$$

$$\text{P.I.} = \frac{1}{D^2 + 5D - 6}(\sin 3x + \cos 2x)$$

$$= \frac{1}{D^2+5D-6}\sin 3x + \frac{1}{D^2+5D-6}\cos 2x$$

$$= \frac{1}{-9+5D-6}\sin 3x + \frac{1}{-4+5D-6}\cos 2x$$

$$= \frac{1}{5D-15}\sin 3x + \frac{1}{5D-10}\cos 2x$$

$$= \frac{(D+3)}{5(D^2-9)}\sin 3x + \frac{(D+2)}{5(D^2-4)}\cos 2x$$

$$= \frac{(D+3)}{5(-9-9)}\sin 3x + \frac{(D+2)}{5(-4-4)}\cos 2x$$

$$= -\frac{1}{90}[3\cos 3x + 3\sin 3x] - \frac{1}{40}[-2\sin 2x + 2\cos 2x]$$

$$= -\frac{1}{30}(\cos 3x + \sin 3x) + \frac{1}{20}(\sin 2x - \cos 2x)$$

The complete solution, is $y = \text{C.F.} + \text{P.I.}$

$$= c_1e^{-6x} + c_2e^{x} - \frac{1}{30}(\cos 3x + \sin 3x) + \frac{1}{20}(\sin 2x - \cos 2x)$$

Q. 3. (b) Solve the following: $\frac{dx}{dt} = 3x + 6y$, $\frac{dy}{dt} = -x - 3y$ with $x(0) = 6$ and $y(0) = -2$.

Ans. Writing D for $\frac{d}{dt}$, the given equations become

$$(D-3)x - 8y = 0 \quad \text{...(1)}$$

$$x + (D+3)y = 0 \quad \text{...(2)}$$

with $x(0) = 6$ and $y(0) = -2$

To eliminate y, operating on both sides of (1) by $(D+3)$ and on both sides of (2) by 8 and adding, we get

$$[(D-3)(D+3)+8]x = 0$$

$$(D^2 - 1)x = 0$$

Its auxiliary equation is, $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$x = \text{C.F.} = c_1e^{-t} + c_2e^{t} \quad \text{...(3)}$$

Now, $\frac{dx}{dt} = -c_1e^{-t} + c_2e^{t}$

$$8y = -4c_1e^{-t} - 2c_2e^{t}, \quad y = -\frac{c_1}{2}e^{-t} - \frac{c_2}{4}e^{t} \quad \ldots(4)$$

Given, $x(0) = 6$ and $y(0) = -2$,

From (3) and (4)

$$c_1 + c_2 = 6 \text{ and } -\frac{c_1}{2} - \frac{c_2}{4} = -2 \Rightarrow c_1 = 2, c_2 = 4$$

$$\therefore \quad x = 2e^{-t} + 4e^{t}, \quad y = -e^{-t} - e^{t}$$

Q. 3. (c) Solve : $x\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - \dfrac{dy}{dx} = 0$.

Ans. $$x\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx} = 0 \quad \ldots(1)$$

put $\dfrac{dy}{dx} = p \Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$

$$(1) \Rightarrow \quad x\frac{dp}{dx} + x\,p^2 - p = 0, \Rightarrow x\frac{dp}{dx} - p = -xp^2$$

$$\left(-\frac{1}{p^2}\right)\frac{dp}{dx} + \frac{1}{x}\left(\frac{1}{p}\right) = 1 \quad \ldots(2)$$

put $\dfrac{1}{p} = v \Rightarrow -\dfrac{1}{p^2}\dfrac{dp}{dx} = \dfrac{dv}{dx}$.

$$(2) \Rightarrow \quad \frac{dv}{dx} + \frac{1}{x}v = 1 \quad \ldots(3)$$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Multiplying (3) by I.F. and integrating, we get

$$v.x = \frac{x^2}{2} + c_1 \Rightarrow v = \frac{x^2 + 2c_1}{2x} \Rightarrow \frac{1}{p} = \frac{x^2 + 2c_1}{2x},$$

$$p = \frac{2x}{x^2 + 2c_1} \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2 + 2c_1} \Rightarrow dy = \frac{2x}{x^2 + 2c_1}dx$$

Again integrating $y = \log(x^2 + 2c_1) + c_2$

Q. 4. (a) Show that $J''_1(x) = -J_1(x) + \dfrac{1}{x}J_2(x)$.

Ans. We have, $xJ'_n = nJ_n - xJ_{n+1} \Rightarrow J'_n = \dfrac{n}{x}J_n - J_{n+1}$

put $n = 1$, $$J'_1 = \frac{1}{x}J_1 - J_2 \quad \ldots(1)$$

Differentiating, $$J''_1 = \frac{1}{x}J'_1 - \frac{1}{x^2}J_1 - J'_2 \quad \ldots(2)$$

By recurrence relation, $xJ'_n = -nJ_n + xJ_{n-1}$

$$xJ'_{n+1} = -(n+1)J_{n+1} + xJ_n$$

$$J'_2 = -\frac{2}{x} J_2 + J_1 \qquad \ldots(3)$$

put (1) and (3) in (2)

$$J''_1 = \frac{1}{x}\left(\frac{1}{x} J_1 - J_2\right) - \frac{1}{x^2} J_1 - \left(-\frac{2}{x} J_2 + J_1\right) = \frac{1}{x^2} J_1 - \frac{1}{x} J_2 - \frac{1}{x^2} J_1 + \frac{2}{x} J_2 - J_1$$

$$J''_1 = -J_1 + \frac{1}{x} J_2$$

Q. 4. (b) Prove that : $x P^1_{n-1}(x) + n P_{n-1}(x) = P^1_n(x)$.

Ans. $xP'_{n-1}(x) + nP_{n-1}(x) = P'_n(x)$

Recurrence relation $(2n+1) P_n = P'_{n+1} - P'_{n-1}$

put $n = n-1$, $(2n-1) P_{n-1} = P'_n - P'_{n-2}$ ...(1)

Recurrence relation, $nP_n = xP'_n - P'_{n-1}$

$$n = n - 1$$

$$(n-1) P_{n-1} = xP'_{n-1} - P'_{n-2} \qquad \ldots(2)$$

Substracting (2) from (1), we get

$$(2n - 1 - n + 1) P_{n-1} = (P'_n - P'_{n-2}) - (x P'_{n-1} - P'_{n-2})$$

$$nP_{n-1} = P'_n - x P'_{n-1}$$

$$x P'_{n-1} + n P_{n-1} = P'_n$$

Q. 4. (c) Evaluate $\int_{-1}^{+1} x^2 P_n^2(x)\, dx$.

Ans. $$I = \int_{-1}^{1} x^2 P_n^2(x)\, dx \qquad \ldots(1)$$

$$(2n+1)\, x\, P_n = (n+1) P_{n+1} + n P_{n-1}$$

$$x P_n = \frac{1}{2n+1}[(n+1) P_{n+1} + n P_{n-1}]$$

$$I = \int_{-1}^{1} \frac{1}{(2n+1)^2}[(n+1) P_{n+1} + n P_{n-1}]^2 dx$$

$$= \frac{1}{(2n+1)^2} \int_{-1}^{1} [(n+1)^2 (P_{n+1})^2 + n^2 P_{n-1}^2 + 2n(n+1) P_{n+1} P_{n-1}]\, dx$$

$$= \frac{1}{(2n+1)^2}\left[(n+1)^2 \int_{-1}^{1} (P_{n+1})^2 dx + n^2 \int_{-1}^{1} (P_{n-1})^2 dx + 2n(n+1) \int_{-1}^{1} P_{n+1} P_{n-1} dx\right]$$

$$= \frac{1}{(2n+1)^2}\left[(n+1)^2 \frac{2}{2(n+1)+1} + n^2 \frac{2}{2(n-1)+1)} + 2n(n+1)\cdot 0\right]$$

$$= \frac{1}{(2n+1)^2}\left[\frac{2(n+1)^2}{(2n+3)} + \frac{2n^2}{(2n-1)}\right]$$

Q. 5. (a) Find the Laplace transform of the following periodic function :

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \frac{\pi}{\omega} \\ 0 \quad , & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases} \qquad \textbf{with period is } \frac{2\pi}{\omega}.$$

$$L[f(t)] = \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \int_0^{2\pi/\omega} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t\, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{\omega e^{-\pi s/\omega} + \omega}{(1-e^{-2\pi s/\omega})(s^2+\omega^2)} = \frac{\omega (1-e^{-\pi s/\omega})}{(1-e^{-\pi s/\omega})(1+e^{-\pi s/\omega})(s^2+\omega^2)}$$

$$= \left(\frac{\omega}{s^2+\omega^2} \right) \left(\frac{1}{1+e^{-\pi s/\omega}} \right)$$

Q. 5. (b) Find the inverse Laplace transform of :

(i) $\frac{s+1}{s^2-6s+25}$ (ii) $\frac{s}{(s^2+4)^2}$

Ans. (i) $L^{-1}\left[\frac{s+1}{s^2-6s+25} \right] = L^{-1}\left[\frac{(s-3)+4}{(s-3)^2+16} \right]$

$$= e^{3t} L^{-1}\left[\frac{s+4}{s^2+16} \right]$$

$$= e^{3t}\left[L^{-1}\left(\frac{s}{s^2+16} \right) + L^{-1}\left(\frac{4}{s^2+16} \right) \right]$$

$$= e^{3t} [\cos 4t + \sin 4t]$$

Ans. (ii) Let $f(t) = L^{-1}\left[\frac{s}{(s^2+4)^2} \right]$

$$\frac{f(t)}{t} = L^{-1}\left[\int_s^\infty \frac{s}{(s^2+4)^2}\, ds \right] = L^{-1}\left[\left\{ -\frac{1}{2(s^2+4)} \right\}_s^\infty \right]$$

$$= L^{-1}\left[\frac{1}{2(s^2+4)} \right] = \frac{1}{2} L^{-1}\left(\frac{1}{s^2+4} \right)$$

$$\frac{f(t)}{t} = \frac{1}{2} \cdot \frac{1}{2} \sin 2t$$

$$f(t) = \frac{t}{4} \sin 2t$$

Q. 5. (c) An alternative emf $E \sin \omega t$ is applied to circuit with an inductance L and a capacitance C in series. Show that the current in the circuit is :

$$\frac{E\omega}{\cdots} (\cos \omega t - \cos nt) \text{ where } n^2 = \frac{1}{LC}.$$

$$L\frac{di}{dt}+\frac{q}{c}=E\sin\omega t$$

$$L\frac{d^2i}{dt^2}+\frac{1}{c}\frac{dq}{dt}=\omega E\cos\omega t$$

$$L\frac{d^2i}{dt^2}+\frac{1}{c}i=\omega E\cos\omega t \qquad \left(\text{Since } i=\frac{dq}{dt}\right)$$

$$\frac{d^2i}{dt^2}+\frac{1}{LC}i=\frac{\omega E}{L}\cos\omega t$$

$$\frac{d^2i}{dt^2}+n^2i=\frac{\omega E}{L}\cos\omega t \qquad \left(\text{Take } n^2=\frac{1}{LC}\right)$$

Auxiliary equation is,

$$m^2+n^2=0 \Rightarrow m=\pm ni$$

C.F. $= c_1\cos nt + c_2\sin nt$,

$$\text{P.I.}=\frac{1}{D^2+n^2}\frac{\omega E}{L}\cos\omega t$$

$$=\frac{E\omega}{L}\frac{1}{-\omega^2+n^2}\cos\omega t \ \text{ if } n^2 \neq -\omega^2=\frac{E\omega}{L(n^2-\omega^2)}\cos\omega t$$

$i=$ C.F. + P.I.

$$i=c_1\cos nt+c_2\sin nt+\frac{E\omega}{L(n^2-\omega^2)}\cos\omega t \qquad \text{...(1)}$$

$$\frac{di}{dt}=-c_1n\sin nt+c_2n\cos nt-\frac{E\omega}{L(n^2-\omega^2)}\sin\omega t \qquad \text{...(2)}$$

Initially $i=0, \frac{di}{dt}=0,$ when $t=0$

$$(1)\Rightarrow, \quad 0=c_1+\frac{E\omega}{(n^2-\omega^2)L}$$

$$c_1=-\frac{E\omega}{(n^2-\omega^2)L}$$

$$(2)\Rightarrow, \quad 0=-c_2n \Rightarrow c_2=0$$

From (1) $i=-\frac{E\omega}{(n^2-\omega^2)L}\cos nt+0+\frac{E\omega}{(n^2-\omega^2)L}\cos\omega t$

$$i=\frac{E\omega}{(n^2-\omega^2)L}(\cos\omega t-\cos nt), \text{ where } n^2=\frac{1}{LC}$$

Q. 6. (a) Solve the partial differential equation $(x^2-yz)p+(y^2-zx)q=z^2-xy$.

Ans. The auxiliary equation are

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$

$$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}=\frac{dz-dx}{(z-x)(x+y+z)}$$

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

On integration, $\log(x - y) = \log(y - z) + \log a$

$$\log\left(\frac{x - y}{y - z}\right) = \log a$$

$$\left(\frac{x - y}{y - z}\right) = a \qquad \ldots(1)$$

Similarly, taking last two ratio, $\left(\frac{y - z}{z - x}\right) = b$ ...(2)

From (1) and (2), the general solution is $f\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0$

Q. 6. (b) Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \, \partial y} = \sin x \cos 2y$.

Ans. The auxiliary equation is, $m^2 - m = 0 \Rightarrow m(m - 1) = 0$

$\Rightarrow \quad m = 0, 1$

$$\text{C.F.} = f_1(y + 0) + f_2(y + x) = f_1(y) + f_2(y + x)$$

$$\text{P.I.} = \frac{1}{D^2 - DD'}(\sin x \cos 2y)$$

$$= \frac{1}{2} \frac{1}{D^2 - DD'}[\sin(x + 2y) + \sin(x - 2y)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\sin(x + 2y) + \frac{1}{D^2 - DD'}\sin(x - 2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1 + 2}\sin(x + 2y) + \frac{1}{-1 - 2}\sin(x - 2y)\right]$$

$$= \frac{1}{2}\sin(x + 2y) - \frac{1}{6}\sin(x - 2y)$$

The complete solution is

$$z = \text{C.F.} + \text{P.I.}$$

$$= f_1(y) + f_2(y + x) + \frac{1}{2}\sin(x + 2y) - \frac{1}{6}\sin(x - 2y)$$

Q. 6. (c) Solve the partial differential equation $(D + 1)(D + D' - 1)z = \sin(x + 2y)$.

Ans. $(D + 1)(D + D' - 1)z = \sin(x + 2y)$

$$\text{C.F.} = e^{-x} f_1(y) + e^{x} f_2(y - x)$$

$$\text{P.I.} = \frac{1}{(D + 1)(D + D' - 1)}\sin(x + 2y)$$

$$= \frac{1}{D^2 + DD' + D' - 1}\sin(x + 2y) = \frac{1}{-1 - 2 + D' - 1}\sin(x + 2y)$$

$$\frac{1}{\ldots}\sin(x + 2y) = \frac{(D' + 4)}{\ldots}\sin(x + 2y)$$

$$= \frac{(D' + 4)}{-4 - 16} \sin(x + 2y) = -\frac{1}{20}[2\cos(x + 2y) + 4\sin(x + 2y)]$$

$$= -\frac{1}{10}[\cos(x + 2y) + 2\sin(x + 2y)]$$

The complete solution is

$$z = \text{C.F.} + \text{P.I.}$$

$$= e^{-x} f_1(y) + e^{x} f_2(y - x) - \frac{1}{10}[\cos(x + 2y) + 2\sin(x + 2y)]$$

Q. 7. (a) Solve the following PDE by method of separation of variable

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u, \text{ with } u(0, y) = 0 \text{ and } \frac{\partial u(0, y)}{\partial x} = 1 + e^{-3y}.$$

Ans.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \quad ...(1)$$

with $u(0, y) = 0, \frac{\partial u}{\partial x}(0, y) = 1 + e^{-3y}$

Let $u = X(x)\, Y(y)$

where X is a function of x only and Y is a function of y only.

$(1) \Rightarrow \quad X''Y = XY' + 2XY$

$$\frac{X''}{X} = \frac{Y'}{Y} + 2 = \text{constant} = -p^2$$

$$\frac{X''}{X} = -p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2 X = 0$$

A.E. is, $m^2 + p^2 = 0 \Rightarrow m = \pm pi$

$$X = c_1 \cos px + c_2 \sin px$$

$$\frac{Y'}{Y} + 2 = -p^2 \Rightarrow \frac{Y'}{Y} = -(p^2 + 2)$$

$$\frac{dY}{Y} = -(p^2 + 2)dy$$

Integrating, $\log Y = -(p^2 + 2)y + \log c_3$

$$Y = c_3 e^{-(p^2 + 2)y}$$

$$u(x, y) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-(p^2 + 2)y} \quad ...(2)$$

$u(0, y) = 0 \Rightarrow c_1 c_3 e^{-(p^2 + 2)y} = 0 \Rightarrow c_1 c_3 = 0$

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px)\, c_3 e^{-(p^2 + 2)y}$$

$$\frac{\partial u}{\partial x}(0, y) = 1 + e^{-3y} \Rightarrow c_2 p\, c_3 e^{-(p^2 + 2)y} = 1 + e^{-3y}$$

$\Rightarrow \quad c_2 c_3 p = 1, \quad -(p^2 + 2) = 0$

and $\quad c_2 c_3 p = 1, \quad -(p^2 + 2) = -3$

$\Rightarrow \quad c_2 c_3 = \frac{1}{,}, \quad p^2 = -2 \Rightarrow p = i\sqrt{2}$

$$c_2c_3 = \frac{1}{i\sqrt{2}}$$

and $$c_2c_3 = \frac{1}{p}, \qquad p^2 = 1 \Rightarrow p = 1$$

$$c_2c_3 = 1$$

From (2), the general solution is

$$u(x, y) = \frac{1}{i\sqrt{2}} \sin(i\sqrt{2} \cdot x) + \sin x \cdot e^{-3y}$$

$$= \frac{1}{i\sqrt{2}} i \sin h(x\sqrt{2}) + e^{-3y} \sin x$$

$$= \frac{1}{\sqrt{2}} \sinh(x\sqrt{2}) + e^{-3y} \sin x$$

Q. 7. (b) Find the temperature $u(x, t)$ in a slab whose ends $x = 0$ and $x = l$ are kept at temperature zero and whose initial temperature $f(x)$ is given by

$$f(x) = \begin{cases} A \text{ when } 0 < x < \frac{l}{2} \\ 0 \text{ when } \frac{l}{2} < x < l \end{cases}$$

Ans. On dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

Boundary conditions are

$$\left.\begin{aligned} u(0, t) = 0 \\ u(l, t) = 0 \end{aligned}\right\} \qquad \ldots(2)$$

Initial condition is

$$u(x, 0) = f(x) = \begin{cases} A, \text{ when } 0 < x < \frac{l}{2} \\ 0, \text{ when } \frac{l}{2} < x < l \end{cases} \qquad \ldots(3)$$

Let $u = XT$

where X is function of x only and T is function of t only.

From (1) and (2), $XT' = C^2X''T$

$$\frac{X''}{X} = \frac{T'}{c^2T} = \text{constant} = -p^2$$

$$\frac{X''}{X} = -p^2 \Rightarrow \frac{d^2x}{dx^2} + p^2x = 0$$

A.E. is $$m^2 + p^2 = 0 \Rightarrow m = \pm pi$$

$$X = c_1 \cos px + c_2 \sin px$$

$$\frac{T'}{c^2T} = -p^2 \Rightarrow \frac{dT}{T} = -c^2p^2dt$$

$$T = c_3 e^{-c^2 p^2 t}$$

Hence, $u(x, t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2 p^2 t}$ ...(5)

using B.C. (2) in (5), we get

$$u(0, t) = 0 \quad \Rightarrow \quad c_1 = 0$$

$$u(l, t) = 0 \quad \Rightarrow \quad c_2 c_3 \sin pl\, e^{-c^2 p^2 t} = 0$$

$$\sin pl = 0 = \sin n\pi$$

$$p = \frac{n\pi}{l}$$

Hence, from (5),

$$u(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 c^2 t}{l^2}}$$

$$= b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 c^2 t}{l^2}}, \text{ where } b_n = c_2 c_3$$

The general solution is

$$u(x, t) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 c^2 t}{l^2}} \qquad ...(6)$$

using I.C. (3) in (6), we get

$$f(x) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} A \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} 0 \cdot \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2A}{l}\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/2}$$

$$= \frac{2A}{l}\left[-\frac{l}{n\pi}\cos\frac{n\pi}{2} + \frac{l}{n\pi}\right]$$

$$= \frac{2A}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$$

Hence, the solution is

$$u(x, t) = \frac{2A}{\pi}\sum_{n=0}^{\infty} \frac{\left(1 - \cos\frac{n\pi}{2}\right)}{n} \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 c^2 t}{l^2}}$$

Q. 7. (c) Solve $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$ with houndary conditions,

(i) v is finite when $r \to 0$ **(ii) $v = \Sigma c_n \cos n\theta$ on $r = a$.**

Ans. $$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0 \quad ...(1)$$

Boundary conditions are

(i) v is finite when $r \to 0$

(ii) $v = \Sigma c_n \cos n\theta$ on $r = a$

Let $$v = RT \quad ...(2)$$

where R is a function of r only and T is a function of θ only

$$r^2 R''T + rR'T + RT'' = 0$$

$$\frac{r^2 R'' + rR'}{R} + \frac{T''}{T} = 0$$

$$\frac{r^2 R'' + rR'}{R} = -\frac{T''}{T} = n^2$$

$\Rightarrow$ $$r^2 R'' + rR' - n^2 R = 0$$

Put $$r = e^z \Rightarrow z = \log r, D \equiv \frac{d}{dz}$$

$$[D(D-1) + D - n^2] R = 0$$

$$(D^2 - n^2) R = 0$$

A.E. is $$m^2 - n^2 = 0 \Rightarrow m = \pm n$$

$$R = c_1 e^{nz} + c_2 e^{-nz} = c_1 r^n + c_2 r^{-n}$$

$$\frac{T''}{T} = -n^2 \Rightarrow \frac{d^2 T}{d\theta^2} + n^2 T = 0$$

A.E. is $$m^2 + n^2 = 0 \Rightarrow m = \pm ni$$

$$T = c_3 \cos n\theta + c_4 \sin n\theta$$

$$v = (c_1 r^n + c_2 r^{-n})(c_3 \cos n\theta + c_4 \sin n\theta)$$

Taking $$c_3 = \cos\alpha, c_4 = -\sin\alpha$$

$$v = (c_1 r^n + c_2 r^{-n}) \cos(n\theta + \alpha) \quad ...(3)$$

v is finite when $r \to 0$, from (3)

$$c_2 = 0$$

(3) $\Rightarrow$ $$v = c_1 r^n \cos(n\theta + \alpha) \quad ...(4)$$

$$v = \Sigma c_n \cos n\theta \text{ on } r = a, \text{ from (4)}$$

$$c_1 = \frac{1}{a^n}, \ \alpha = 0$$

Hence, from (4) $$v = \left(\frac{r}{a}\right)^n \cos n\theta$$